THREE-SPHERE INEQUALITIES WITH PARTIAL DATA, CARLEMAN'S ESTIMATES, AND THEIR APPLICATIONS

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Three-sphere inequality plays an important role in the study of the stability and the unique continuation principle for elliptic equations. A famous example is the Hadamard three sphere inequality for holomorphic functions v in B_R:

$$\|v\|_{L^\infty(\partial B_{R_2})} \le \|v\|^{\alpha}_{L^\infty(\partial B_{R_1})} \|v\|^{1-\alpha}_{L^\infty(\partial B_{r_3})} \tag{0.1}$$

for all $0 < R_1 < R_2 < R_3 < R$ where

$$\alpha = \log\Big(\frac{R_3}{R_2}\Big) \Big/ \log\Big(\frac{R_3}{R_1}\Big).$$

This type of inequality is related and can be derived from Carleman's estimates. The objective of the course is to give an introduction on three-sphere inequalities, Carleman's estimates. Classical and new aspects are discussed. Their applications on metamaterials and controls are given.

The program of the course is described below:

• First part: Introduction. In this part, I will discuss several simple proofs of three spheres inequality for the case $\Delta u = 0$ based on separation of variables and the frequency function.

• Second part: Three sphere-inequality for elliptic equations and their applications in metamaterials. In this part, I discuss the proof of a three sphere inequality for the following equation

$$\operatorname{div}(M\nabla v) + b \cdot \nabla v + cv = 0,$$

under the assumption that M is uniformly elliptic and smooth, and b and c are bounded. The applications for cloaking and superlensing using negative index materials are then presented.

• Third part: Three sphere inequality with partial information and their application in metamaterials. In this part, I will discuss a three sphere inequality with partial information for the equation

$$\Delta u = 0$$

in two dimensions and its application for metamaterials.

• Fourth part: In this part, I will present Carleman's estimates for parabolic equations of the type Lebeau-Robbiano and discuss its connection with the null-controllability of the heat equation. I also deal with the question how the cost function depends on the the location of the support for the heat equation.