

SPECTRAL STABILITY OF SMALL-AMPLITUDE SHOCK PROFILES FOR DISSIPATIVE SYMMETRIC HYPERBOLIC-PARABOLIC SYSTEMS

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ABSTRACT. We prove one-dimensional spectral stability of small amplitude shock profiles for degenerate viscosity conservation laws that are dissipative symmetric hyperbolic-parabolic in the sense of Kawashima [Ka.1], generalizing well-known results of Matsumura and Nishihara [MN] and of Kawashima, Matsumura, and Nishihara [KM,KMN] for the case of compressible gas dynamics. The proof follows an approach different from that of [MN,KM,KMN], instead combining the weighted energy method of Goodman [Go.2] with the auxiliary derivative estimates of Kawashima [Ka.1].

Section 1. Introduction

Consider a one-dimensional system of conservation laws

$$(1.1)_{\text{cons}} \quad u_t + f(u)_x = (B(u)u_x)_x,$$

$u, f \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$, $f, B \in C^2$, that is of *symmetric hyperbolic-parabolic type* in the sense of Kawashima [Ka.1,KSy] in some neighborhood \mathcal{U} of a particular base point u_* , i.e.:

(+) For all $u \in \mathcal{U}$, there exists a symmetrizer $A^0(u)$, symmetric and positive definite, such that $A^0(u)A(u)$ is symmetric, $A(u) := df(u)$, and $A^0(u)B(u)$ is symmetric, positive semidefinite.

Note that (+) implies that also $(A^0)^{1/2}A(A^0)^{-1/2}$ is symmetric, and so the spectrum of A is real and semi-simple, i.e. the first-order part of (1.1) is (nonstrictly) hyperbolic. In addition, suppose that there hold Kawashima's conditions of *dissipativity*:

(++) For $u \in \mathcal{U}$, there is no eigenvector of $A(u)$ lying in the kernel of $B(u)$;

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and *block structure*:

(+++)
The right kernel of $B(u)$ is independent of u .

These properties are enjoyed by many of the equations of continuum mechanics, in particular the equations of compressible gas dynamics and magnetohydrodynamics. In such applications, the matrix $B(u)$ is usually singular, i.e. (1.1) is incompletely parabolic. The significance of $(+)-(+++)$ is that behavior is nonetheless similar in many ways to what would be seen in the parabolic case; for example, the “genuine coupling” of hyperbolic and parabolic effects embodied in condition $(++)$ has been shown in several contexts to imply time-asymptotic smoothing and large-time behavior similar to that of the strictly parabolic case [MNi,Ka.1–2,Ze,LZe,Ho,HoZ.1–2].

In particular, at least for small amplitude waves, conditions $(+)-(+++)$ imply that the viscosity B is sufficiently regularizing to “smooth” discontinuous traveling wave solutions, or “shock waves,”

$$(1.2)_{\text{shock}} \quad u(x, t) = \bar{u}(x - st) := \begin{cases} u_- & x - st < 0, \\ u_+ & x - st \geq 0, \end{cases}$$

of the corresponding hyperbolic equations

$$(1.3)_{\text{hyp}} \quad u_t + f(u)_x = 0,$$

yielding instead *smooth* traveling wave solutions

$$(1.4)_{\text{vshock}} \quad u = \bar{u}(x - st); \quad \lim_{z \rightarrow \pm\infty} \bar{u}(z) = u_{\pm},$$

or “viscous shock profiles.” This fact, well-known in the context of gas dynamics [We,Gi], was recently established by Freistühler [Fre] for general Kawashima class systems, using a variation on the center manifold argument of Majda and Pego [MP] in the strictly parabolic case.

More precisely, let

$$a_1(u) \leq \cdots \leq a_n(u)$$

denote the eigenvalues of $A = df(u)$, $r_j(u)$ and $l_j(u)$ a smooth choice of associated right and left eigenvectors, $l_j \cdot r_k = \delta_k^j$, and assume at the base point u_* that:

(H1) The p th characteristic field is of multiplicity one, i.e. $a_p(u_*)$ is a simple eigenvalue of $A(u_*)$.

(H2) The p th characteristic field is genuinely nonlinear, i.e. $\nabla a_p \cdot r_p(u_*) \neq 0$.

Then, we have:

Proposition 1.^{freist1} ([Fre]). *Let $(+)-(+++)$ and (H1)–(H2) hold. Then, for left and right states u_{\pm} lying within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_* , and speeds s lying within a sufficiently small neighborhood of $a_p(u_*)$, there exists a*

viscous profile (1.4) that is “local” in the sense that the image of $\bar{u}(\cdot)$ lies entirely within \mathcal{V} if and only if the triple (u_-, u_+, s) satisfies both the Rankine–Hugoniot relations:

$$(RH) \quad s[u] = [f],$$

and the Lax characteristic conditions for a p -shock:

$$(L) \quad a_p(u_-) > s > a_p(u_+); \quad \text{sgn} \ (a_j(u_-) - s) = \text{sgn} \ (a_j(u_+) - s) \neq 0 \text{ for } j \neq p.$$

(Note: The structure theorem of Lax [La, Sm] implies that (RH), always a necessary condition for existence of profiles, holds for $u_{\pm} \in \mathcal{V}$ only if s lies near some $a_j(u_*)$; thus, the restriction on speed s is only the assumption that the triple (u_-, u_+, s) be associated with the p th and not some other characteristic field).

The purpose of the present paper is to establish the spectral stability of such shock profiles as solutions of the “viscous” equations (1.1), under the same hypotheses (+)–(+++), (H1)–(H2). Take without loss of generality $s = 0$, so that $u = \bar{u}(x)$ becomes a stationary solution. Then, the linearized equations of (1.1) about \bar{u} take the form

$$(1.5)_{\text{linearized}} \quad v_t = Lv := -[(A + E)v]_x + (Bv')',$$

where

$$(1.6)_{AB} \quad B := B(\bar{u}), \quad A := df(\bar{u}),$$

and

$$(1.7)_E \quad Ev := -(dBv)\bar{u}_x.$$

Definition 1.^{stable}2. We call the profile $\bar{u}(\cdot)$ *spectrally stable* if the linearized operator L about the wave has no spectrum in the closed unstable complex half-plane $\{\lambda : \text{Re } \lambda \geq 0\}$ except at the origin, $\lambda = 0$. (Recall, [Sat], that $\lambda = 0$ is always in the spectrum of L , since $L\bar{u}_x = 0$ by direct calculation/differentiation of the traveling wave ODE).

Our main result is then:

Theorem 1.^{main}3. *Let (+)–(+++) and (H1)–(H2) hold, and let $\bar{u}(x - st)$ be a viscous shock solution such that the profile $\{\bar{u}(z)\}$ lies entirely within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_* , and the speed s lies within a sufficiently small neighborhood of $a_p(u_*)$. Then, \bar{u} is spectrally stable, in the sense of Definition 1.2 above.*

Spectral stability is in general a weaker notion than linearized stability, which in turn is weaker than nonlinear stability. However, in the present context, we

expect that these three notions can be shown to be equivalent, through the general program set out in [ZH,MZ]; in any case, spectral stability is an essential initial step in this program.

As pointed out in [ZH], spectral stability is implied by linearized stability with respect to zero-mass perturbations; indeed, in most cases, an argument for spectral stability can often be translated directly to a proof of zero-mass stability [Z.1–2]. Thus, Theorem 1.3 may be viewed as a generalization of the zero-mass results obtained early on by Matsumura–Nishihara [MN] and Kawashima–Matsumura–Nishihara [KM,KMN] for small-amplitude shocks of the equations of compressible gas dynamics. It can also be viewed as a generalization of the corresponding result of Goodman [Go.1–2] for small-amplitude shocks of general, strictly parabolic systems, which appeared at roughly the same time.

Interestingly, these two apparently similar results proceed by rather different arguments. Indeed, though it seems natural to conjecture that the results of [MN,KM,KMN] should extend to general Kawashima class systems, we do not see an obvious way to extend the approach of [MN,KM,KMN] to more general systems. We proceed here, instead, by adapting the weighted energy method of Goodman [Go.2] to the degenerate viscosity case, thus achieving a unified approach to the degenerate and the strictly parabolic viscosity case.

The structure of our argument is straightforward: Since Goodman’s approach involves coordinate changes not respecting the spectral structure of A^0B , the resulting diffusion term may in fact be *indefinite*, yielding unfavorable energy estimates in certain modes. However, the extent of deviation from semidefinite positivity is small on the order of the shock amplitude, and so the resulting bad H^1 term in the energy estimate can be controlled by higher order energy estimates of the type described by Kawashima [Ka.1]. An interesting aspect of the analysis is that here, in contrast to [Ka.1], the approach of Kawashima is applied to perturbations of a *nonconstant* background solution, confirming the flexibility of the method.

We remark, finally, that the assumption of genuine nonlinearity (H2) is not needed either for the existence or the stability result, but is made only to simplify the discussion. Though we stated above only the restriction to the genuinely nonlinear case, existence was in fact treated for the general (nongenuinely nonlinear) case in [Fre]. Likewise, to extend our stability argument to the general case, one has only to substitute for the “Goodman-type” weighted energy estimate in Section 5, the variation introduced by Fries [Fri.1–2] to treat the nongenuinely nonlinear case for strictly parabolic viscosities; for, at this point in the argument, the situation is reduced essentially to that of the strictly parabolic case. We suspect, further, that $ReA^0B \geq 0$ can be substituted in (+) for the symmetric, positive semidefinite assumption on A^0B , in both the existence and stability theory, with little change in the arguments.

Section 2. Preliminaries

We begin by collecting some needed, known results.

Lemma 2.^{pbound}**1.** *Let $(+)-(+++)$ and $(H1)-(H2)$ hold, and let $\bar{u}(x - st)$ be a viscous shock solution such that the profile $\{\bar{u}(z)\}$ lies entirely within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_* , and the speed s lies within a sufficiently small neighborhood of $a_p(u_*)$. Then, letting $\varepsilon := |u_+ - u_-|$ denote shock strength, and $\delta := \max |u_{\pm} - u_*|$ the distance from base point u_* , we have bounds*

$$(2.1)_{a4.15} \quad \begin{aligned} \bar{u}' &= \mathcal{O}(\varepsilon^2) e^{-\theta \varepsilon |x|} (r_p(u_*) + \mathcal{O}(\delta)) \\ \bar{u}'' &= \mathcal{O}(\varepsilon^3) e^{-\theta \varepsilon |x|} \end{aligned}$$

and

$$(2.2)_{a4.16} \quad \begin{aligned} \bar{a}'_j &= \mathcal{O}(|\bar{u}'|), \\ \bar{a}''_j &= \mathcal{O}(|\bar{u}''| + |\bar{u}'|^2) = o(|\bar{u}'|), \end{aligned}$$

with, moreover,

$$(2.3)_{pbound} \quad \bar{a}'_p \leq -\theta |\bar{u}'|$$

for some uniform constant $\theta > 0$.

Proof. Though the bounds (2.1)–(2.2) are not explicitly stated in [Fre], they follow immediately from the detailed description of center manifold dynamics obtained in the proof, exactly as in the strictly parabolic case [MP]. Monotonicity, (2.3) follows readily from (2.1) and genuine nonlinearity, (H2), together with the Lax characteristic condition (L). ■

Lemma 2.^{skew}**2** ([SK]). *Assuming condition $(+)$, condition $(++)$ is equivalent to either of:*

(K1) *For each $u \in \mathcal{U}$, there exists a skew-symmetric matrix $K(u)$ such that*

$$(2.4)_{skew} \quad \operatorname{Re} (KA + A^0 B)(u) \geq \theta > 0,$$

A^0 as in $(+)$.

(K2) *For some $\theta > 0$, there holds*

$$(2.5)_{symbol} \quad \operatorname{Re} \sigma(-i\xi A(u) - |\xi|^2 B(u)) \leq -\theta |\xi|^2 / (1 + |\xi|^2),$$

for all $\xi \in \mathbb{R}$.

Proof. For the proof of these and other useful equivalent formulations of $(++)$, see [SK]. ■

Lemma 2.^{noess}3. *Given $(+)-(+++)$, the linearized operator L described in (1.5) has no essential spectrum in $\{\lambda : \operatorname{Re} \lambda \geq 0\} \setminus \{0\}$.*

Proof. This follows by a standard argument of Henry [He], once we establish linearized stability of the constant solutions $u \equiv u_{\pm}$; see [GZ,ZH,ZS,Z.1–2] for further details. But, linearized stability follows immediately from (K2). ■

Lemma 2.3 reduces the question of spectral stability to the existence or nonexistence of *point spectrum* in the deleted unstable half-plane

$$\{\lambda : \operatorname{Re} \lambda \geq 0\} \setminus \{0\},$$

an ODE question. From (1.5), we obtain the linearized eigenvalue equation

$$(2.6)_{\text{evalue}} \quad \lambda w = Lw = -[(A + E)w]' + (Bw')',$$

where “ \prime ” denotes d/dx . What we must show is that there exist no solutions of (2.6) decaying at $\pm\infty$ on the deleted unstable half-plane.

Following [Go.1–2,MN,KM,KMN], consider now the “integrated” equations

$$(2.7)_{\text{ilinearized}} \quad V_t = \mathcal{L}V := -(A + E)V_x + BV_{xx}$$

governing evolution of the integrated variable

$$(2.8)_{\text{v}} \quad V(x) := \int_{-\infty}^x v(y)dy,$$

and the associated eigenvalue equation

$$(2.9)_{\text{intvalue}} \quad \lambda W = \mathcal{L}W = -(A + E)W' + BW''.$$

Lemma 2.^{intvalue}4. *Given $(+)-(+++)$, the spectra of L and the “integrated operator” \mathcal{L} agree on the deleted unstable half-plane $\{\lambda : \operatorname{Re} \lambda \geq 0\} \setminus \{0\}$.*

Proof. By standard considerations related to those of Lemma 2.3, we have on $\{\lambda : \operatorname{Re} \lambda \geq 0\}$ that that bounded solutions of either (2.6) or (2.9) in fact decay exponentially in up to two derivatives as $x \rightarrow \pm\infty$, see e.g. [GZ,ZH,ZS,Z.1–2]. Thus, we find immediately that $\sigma(\mathcal{L}) \subset \sigma(L)$, by differentiation of (2.9). On the other hand, suppose that there exists an eigenvalue $\lambda \neq 0$ of L , $\operatorname{Re} \lambda \geq 0$, with corresponding eigenfunction $w \in L^2$. Integrating (2.6) from ∞ to $+\infty$ thus yields

$$(2.10)_{\text{a4.4}} \quad \lambda \int w = 0,$$

hence the “integrated variable”

$$(2.11)_{\text{a4.5}} \quad W(x) := \int_{-\infty}^x w(y)dy$$

lies also in L^2 , and decays exponentially to zero at $\pm\infty$ in up to one derivative. Moreover, it satisfies (2.9), by integration of (2.6), hence λ is an eigenvalue of \mathcal{L} as well. ■

Similarly as in [Go.1-2,MN,KM,KMN], the “integrated operator” \mathcal{L} can be expected to yield more favorable energy estimates, since it is presumably *uniformly* stable with regard to point spectrum, having no eigenvalue at $\lambda = 0$, or indeed in a neighborhood of the closed unstable half-plane.

Following the philosophy of [ZH,Z.1], we will carry out energy estimates in the frequency domain, that is, on the eigenvalue equations (2.6), (2.9) rather than the evolutionary equations (1.5), (2.7). For convenience of the reader, we give here the elementary computation that plays in the spectral, complex-valued context the role played by Friedrich’s-type estimates for real-valued time-evolutionary systems with symmetric coefficients [Fr]. Hereafter, let $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ denote the standard complex L^2 norm and inner product, $|\cdot|$ and “ \cdot ” the complex vector norm and inner product, and $\int f$ the integral $\int_{-\infty}^{+\infty} f(x)dx$.

Lemma 2.^{sym}5. *Let $f(x) \in \mathbb{C}^n$ be an H^1 , complex vector-valued function, and $H(x) \in \mathbb{C}^{n \times n}$ a Hermitian, C^1 complex matrix-valued function. Then,*

$$(2.12)_{\text{hermitian}} \quad \operatorname{Re} \langle f, H f' \rangle = -\operatorname{Re} \langle f, (H f)' \rangle = -(1/2) \langle f, H' f \rangle,$$

where “ \prime ” as usual denotes d/dx . Likewise, if $K(x) \in \mathbb{C}^{n \times n}$ is an anti-Hermitian C^1 complex matrix-valued function, then

$$(2.13)_{\text{antihermitian}} \quad \operatorname{Im} \langle f, K f' \rangle = -\operatorname{Im} \langle f, (K f)' \rangle = -(1/2) \langle f, K' f \rangle.$$

Proof. The first equality in (2.12) follows upon integration by parts. Likewise, integrating by parts, we have

$$\begin{aligned} \operatorname{Re} \langle f, H f' \rangle &:= (1/2)(\langle f, H f' \rangle + \langle H f', f \rangle) \\ &= (1/2)(\langle f, H f' \rangle + \langle f', H f \rangle) \\ &= (1/2)(\langle f, H f' \rangle - \langle f, (H f)' \rangle) \\ &= -(1/2) \langle f, H' f \rangle, \end{aligned}$$

verifying the second equality. Setting $H := -iK$, we obtain (2.13) from (2.12). ■

Section 3. Basic Energy Estimates.

We first derive standard, “Friedrichs-type” estimates for (2.6), (2.9) [Fr].

Lemma 3.^{friedrichs} **1.** *Suppose that λ is an eigenvalue of L, \mathcal{L} , with $\operatorname{Re} \lambda \geq 0, \lambda \neq 0$. Then, there hold estimates*

$$(3.1)_{\text{Ia}} \quad \operatorname{Re} \lambda \|W\|^2 + \|BW'\|^2 \leq C \int |\bar{u}'| |W|^2,$$

$$(3.2)_{\text{Ic}} \quad |\operatorname{Im} \lambda| \int |\bar{u}'| |W|^2 \leq C \int |\bar{u}'| (\eta |W|^2 + \eta^{-1} |W'|^2),$$

and

$$(3.3)_{\text{Ib}} \quad \operatorname{Re} \lambda \|w\|^2 + \|Bw'\|^2 \leq C \int |\bar{u}'| |w|^2,$$

for some constant $C > 0$, any $\eta > 0$.

Proof. From (1.6)–(1.7), we have

$$(3.4)_{\text{ests}} \quad |A'|, |E| = O(|\bar{u}'|).$$

Similarly, by $(++)$, the block structure assumption $(+++)$, and (1.7), we have

$$(3.5)_{\text{pos}} \quad v \cdot (A^0 B v) \geq |Bv|^2 / C,$$

$$(3.6)_{\text{Best}} \quad |(A^0 B)' v| \leq C |\bar{u}'| |Bv|,$$

$$(3.7)_{\text{EBest}} \quad |A^0 E v| \leq C |\bar{u}'| |Bv|,$$

for any vector v , for some constant $C > 0$.

Taking the real part of the L^2 inner product of $A^0 W$ against (2.9), applying (1.7) and (2.12), and integrating the viscous (second-order) term by parts, we thus obtain

$$\begin{aligned} \operatorname{Re} \lambda \langle W, A^0 W \rangle &= \operatorname{Re} \langle W, A^0 B W'' \rangle - \operatorname{Re} \langle W, A^0 E W' \rangle + (1/2) \langle W, (A^0 A)' W \rangle \\ &= -\langle W', A^0 B W' \rangle - \operatorname{Re} \langle W, [(A^0 B)' - A^0 E] W' \rangle \\ &\quad + (1/2) \langle W, (A^0 A)' W \rangle \\ &= -\langle W', A^0 B W' \rangle + \int O(|\bar{u}'|) (|B W'|^2 + |W|^2), \end{aligned}$$

and, rearranging, and absorbing $O(\int |\bar{u}'| |B W'|^2) = O(\varepsilon \|B W'\|^2)$ into the favorable term $-\langle W', A^0 B W' \rangle \leq -\|B W'\|^2 / C$, we obtain the claimed inequality (3.1). Inequalities (3.2) and (3.3) follow similarly, with the parameter η arising in (3.2) by an application of Young's inequality. (Note the appearance of multiplier $|\bar{u}'|$ in the lefthand side of (3.2)). ■

Corollary 3.^{smalllambda}**2.** *Suppose that λ is an eigenvalue of L, \mathcal{L} , with $\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$. Then, $|\operatorname{Re} \lambda| \leq C\varepsilon^2$, for some constant $C > 0$.*

Proof. Otherwise, the righthand side of (3.1) can be absorbed in the term $\operatorname{Re} \lambda \|W\|^2$, since $|\bar{u}'| \leq C\varepsilon^2$, by (2.1). But, this implies $W \equiv 0$, a contradiction. ■

Section 4. Derivative Estimate.

Next, we carry out a nonstandard derivative estimate of the type formalized by Kawashima [Ka.1]. The origin of this approach goes back to [K,MNi] in the context of gas dynamics; see, e.g., [HoZ.1] for further discussion/references.

Lemma 4.^{deriv1}**1.** *Suppose that λ is an eigenvalue of L, \mathcal{L} , with $\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$. Then,*

$$(4.1)_{\text{Kest}} \quad \|W'\|^2 \leq C(|\operatorname{Re} \lambda| \eta \|W\|^2 + \int |\bar{u}'| |W|^2 + \|BW''\|^2/\eta),$$

for some constant $C > 0$ and $\eta > 0$, ε^2/η sufficiently small.

Proof. Taking the real part of the L^2 inner product of W' against K times (2.9), where K is as in (K1), applying (2.13), and using Young's inequality repeatedly, we obtain

$$\begin{aligned} \operatorname{Re} (\langle W', KAW' \rangle) &= \operatorname{Re} (-\lambda \langle W', KW \rangle - \langle W', KEW' \rangle + \langle W', KBW'' \rangle) \\ &\leq |\operatorname{Re} \lambda| \langle |W'|, |KW| \rangle + |\operatorname{Im} \lambda| \langle |W|, |K'W| \rangle \\ &\quad + \langle |W'|, |KEW'| \rangle + \langle |W'|, |KBW''| \rangle \\ &\leq C[|\operatorname{Re} \lambda| (\|W'\|^2/\eta + \eta \|W\|^2) + |\operatorname{Im} \lambda| \int |\bar{u}'| |W|^2 \\ &\quad + (\varepsilon^2 \|W'\|^2 + (\eta \|W'\|^2 + \|BW''\|^2/\eta))]. \end{aligned}$$

Recalling that $|\operatorname{Re} \lambda| \leq C\varepsilon^2$ by Corollary 3.2, and $\operatorname{Re} \langle W', KAW' \rangle \geq \|W'\|^2/C$ by (K1), we find for $\eta, \varepsilon^2/\eta$ sufficiently small that the terms $|\operatorname{Re} \lambda| \|W'\|^2/\eta$, $C\varepsilon \|W'\|^2$, and $C\eta \|W'\|^2$ can be absorbed in the left hand side, yielding

$$(4.2)_{\text{prelim}} \quad \|W'\|^2 \leq C(|\operatorname{Re} \lambda| \eta \|W\|^2 |\operatorname{Im} \lambda| + \int |\bar{u}'| |W|^2 + \|BW''\|^2/\eta).$$

Applying bound (3.2) and recalling that $|\bar{u}'| \leq C\varepsilon^2$, we find for $\eta, \varepsilon^2/\eta$ sufficiently small that the term $C|\operatorname{Im} \lambda|$ on the righthand side may be absorbed in the lefthand side and $C \int |\bar{u}'| |W|^2$, giving the result. ■

Corollary 4.^{deriv2}**2.** *Suppose that λ is an eigenvalue of L , \mathcal{L} , with $\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$. Then,*

$$(4.3)_{\text{semifinal}} \quad \lambda \|W\|^2 + \|W'\|^2 + \|BW''\|^2 \leq C \int |\bar{u}'| |W|^2$$

for some constant $C > 0$, for all ε sufficiently small.

Proof. Adding C times (3.1), C/η times (3.3), and (4.1), with $C > 0$ sufficiently large, and η sufficiently small, we obtain the result. (Recall that $BW'' = Bw'$). ■

Section 5. Weighted Energy Estimate.

At this point, we have reduced the problem essentially to the situation of the strictly parabolic case. Evidently, the main issue here, as there, is to control the term $C \int |\bar{u}'| |W|^2$ on the righthand side of (4.3). This we can accomplish using the weighted energy method of Goodman [Go.2] with a bit of extra care.

Lemma 5.^{blockdiag}**1.** *Assuming $(+)-(+++)$ and $(H1)-(H2)$, there exist smooth, real matrix-valued functions $\tilde{R}(u)$, $\tilde{L}(u)$, $\tilde{L}\tilde{R} = I$, such that*

$$(5.1)_{\text{blockdiag}} \quad \tilde{L}A\tilde{R} = \begin{pmatrix} A_- & 0 & 0 \\ 0 & a_p & 0 \\ 0 & 0 & A_+ \end{pmatrix},$$

where $A_- \leq a_- < 0$ and $A_+ \geq a_+ > 0$ are symmetric, and

$$(5.2)_{\text{semipos}} \quad \tilde{L}B\tilde{R} \geq 0$$

is symmetric, positive semi-definite.

Proof. As we observed in the introduction, $(+)$ implies that $(A^0)^{1/2}A(A^0)^{-1/2}$ is symmetric, and likewise $(A^0)^{1/2}B(A^0)^{-1/2}$ is symmetric, positive semidefinite. By $(H1)$, there is spectral separation between eigenvalue a_p and the positive and negative spectra of matrix $(A^0)^{1/2}A(A^0)^{-1/2}$, hence it can be block diagonalized by a real, orthogonal transformation $O(A^0)^{1/2}A(A^0)^{-1/2}O^t$, $O^t = O^{-1}$, which likewise preserves symmetry, and semidefinite positivity of $(A^0)^{1/2}B(A^0)^{-1/2}$. Setting $\tilde{R} = (A^0)^{-1/2}O^t$, $\tilde{L} = O(A^0)^{1/2}$, we are done. ■

Lemma 5.^{special}**2.** *Assuming $(+)-(+++)$ and $(H1)-(H2)$, there exist smooth, real matrix-valued functions $R(u)$, $L(u)$, $LR = I$, such that*

$$(5.3)_{\text{blockdiag}} \quad LAR = \begin{pmatrix} A_- & 0 & 0 \\ 0 & a_p & 0 \\ 0 & 0 & A_+ \end{pmatrix},$$

where $A_- \leq a_- < 0$ and $A_+ \geq a_+ > 0$ are symmetric,

$$(5.4)_{\text{pp}} \quad (LR')_{pp} = (L'R)_{pp} = 0,$$

and

$$(5.5)_{\text{approxpos}} \quad \text{Re } LBR \geq -C\varepsilon$$

for some constant $C > 0$.

Proof. Set $R := \Gamma \tilde{R}$, $L := \Gamma^{-1} \tilde{L}$, with

$$(5.6)_{\text{Gamma}} \quad \Gamma := \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I_{n-p} \end{pmatrix},$$

and define γ by the linear ODE

$$(5.7)_{\text{ode}} \quad \gamma' = -\tilde{l}_p \tilde{r}_p' \gamma; \quad \gamma(0) = 1,$$

where \tilde{l}_p , \tilde{r}_p denote the p th row and column, respectively, of \tilde{L} , \tilde{R} . Clearly, L and R still block-diagonalize A in the manner claimed, while

$$\begin{aligned} (LR')_{pp} &= \gamma^{-1} \tilde{l}_p (\gamma \tilde{r}_p)' = \gamma^{-1} \tilde{l}_p (\gamma' \tilde{r}_p + \gamma \tilde{r}_p') \\ &= \gamma^{-1} (\gamma' + \gamma (\tilde{l}_p \tilde{r}_p')) = 0, \end{aligned}$$

by (5.7). On the other hand, $|r_p'| \leq C|\bar{u}'|$, whence we obtain by direct integration of (5.7) the bound

$$\begin{aligned} \gamma(x) &= e^{\int_0^x -\ell_p r_p'} = e^{O(\int_{-\infty}^{+\infty} |\bar{u}'|)} \\ &= e^{O(\varepsilon)} = 1 + O(\varepsilon), \end{aligned}$$

yielding bound (5.5) by (5.2) and continuity. ■

Lemma 5.^{goodman}**3.** *Let there hold (+)-(+++)³ and (H1)-(H2), and suppose that λ is an eigenvalue of L , \mathcal{L} , with $\text{Re } \lambda \geq 0$, $\lambda \neq 0$. Then,*

$$(5.8)_{\text{good}} \quad \text{Re } \lambda \|W\|^2 + \int |\bar{u}'| |W|^2 \leq C\varepsilon \|W'\|^2$$

for some constant $C > 0$, for all ε sufficiently small.

Proof. By the construction described above, we have, clearly:

$$(5.9)_{\text{a4.19}} \quad \begin{aligned} |L'|, |R'| &= \mathcal{O}(\bar{u}'), \\ |L''|, |R''| &= \mathcal{O}(|\bar{u}''| + |\bar{u}'|^2). \end{aligned}$$

Setting $Z := LW$, and left multiplying (2.9) by L , we thus obtain

$$(5.10)_{a4.20} \quad \lambda Z + (\bar{A} + \bar{E})Z' + \bar{M}Z = (\bar{B}Z)'$$

where $\bar{A} := LAR$ is as in (5.3), $\bar{B} := LBR > -C\varepsilon$, \bar{E} defined by

$$\bar{E}v := LBR'v - L'BRv + L(dB\bar{u}_x)Rv - L(dBRv)\bar{u}_x$$

satisfies

$$(5.11)_{a4.21} \quad \bar{E} = \mathcal{O}(|\bar{u}'|), \quad \bar{E}' = \mathcal{O}(|\bar{u}''| + |\bar{u}'|^2) = \mathcal{O}(\varepsilon|\bar{u}'|),$$

and \bar{M} defined by

$$\bar{M}v := \bar{A}LR'v + L(dB\bar{u}_x)R'v - L(dBR'v)\bar{u}_x - L(BR')'v$$

satisfies

$$(5.12)_{a4.22} \quad |\bar{M}| = \mathcal{O}(\bar{u}'), \quad |\bar{M}_{pp}| = \mathcal{O}(|\bar{u}''| + |\bar{u}'|^2) = \mathcal{O}(\varepsilon|\bar{u}'|),$$

the second estimate following by normalization (5.4). Clearly, to establish (5.8), it is sufficient to establish the corresponding result in Z coordinates.

Following [Go.2], define weight $\alpha_p \equiv 1$, and define weights α_{\pm} by ODE

$$(5.13)_{a4.23} \quad \alpha'_{\pm} = -C|\bar{u}'|\alpha/a_{\pm}, \quad \alpha_{\pm}(0) := 1,$$

whence

$$(5.14)_{a4.24} \quad \begin{aligned} \alpha_{\pm}(x) &= e^{\int_0^x C|\bar{u}'|/a_{\pm}} = 1 + \mathcal{O}(C \int_{-\infty}^{\infty} |\bar{u}'|) \\ &= 1 + \mathcal{O}(C\varepsilon) = \mathcal{O}(1), \end{aligned}$$

$$(5.15)_{a4.25} \quad \alpha'_j = \mathcal{O}(|\bar{u}'|), \quad j = -, p, +.$$

Here, C is a sufficiently large constant to be chosen later, and ε is so small that $\mathcal{O}(C\varepsilon) < 1$. Set $\alpha := \text{diag}\{\alpha_j\}$.

Now, take the real part of the complex L^2 inner product of αZ with (5.10), to obtain the energy estimate (after integration by parts)

$$(5.16)_{a4.26} \quad \begin{aligned} \text{Re } \lambda \sum \int \alpha_j |Z_j|^2 - \sum \langle Z_j, (a_j \alpha_j)' Z_j \rangle + \text{Re } \int \langle Z', \alpha \bar{B} Z' \rangle = \\ \text{Re } \int \langle Z, \alpha \bar{M} Z \rangle - \text{Re } \int \langle \alpha Z, \bar{E} Z' \rangle - \text{Re } \int \langle \alpha' Z, \bar{B} Z' \rangle, \end{aligned}$$

where j is summed over $-, p, +$, and $Z =: (Z_-, Z_p, Z_+)^t$. Noting that

$$\begin{cases} (\alpha_p \bar{a}_1)' = \bar{a}'_p < -\theta |\bar{u}'|, \\ (\alpha_j \bar{a}_j)' = \alpha'_j \bar{a}_j + \alpha_j \bar{a}'_j \\ \quad < -C\theta |\bar{u}'| \text{ for } j \neq p, \end{cases}$$

where C may be chosen arbitrarily large, and that $\operatorname{Re} \alpha \bar{B} > -C\varepsilon$ by continuity, for ε sufficiently small, and using estimates (5.11)–(5.12) to absorb all terms in the righthand side, we obtain the result. More precisely, we have used Young's inequality to bound the second and third terms on the righthand side of (5.16) by

$$\begin{aligned} C \int |\bar{u}'| |Z| |Z'| &\leq \frac{C}{2} \left(\int |\bar{u}'|^{3/2} |Z|^2 + \int |\bar{u}'|^{1/2} |Z'|^2 \right) \\ &\leq \frac{C}{2} \left(\varepsilon \int |\bar{u}'| |Z|^2 + \varepsilon \int |Z'|^2 \right), \end{aligned}$$

a contribution that is clearly absorbable on the lefthand side. The first term on the righthand side is bounded by

$$C_2(\varepsilon \int |\bar{u}'| |Z_p|^2 + \int |\bar{u}'| |Z_\pm|^2),$$

where C_2 is some fixed constant, hence it is also absorbable. ■

Section 6. Proof of Theorem 1.3.

The proof of Theorem 1.3 is now straightforward. Adding $C\varepsilon$ times (4.3) to (5.8), we obtain

$$(6.1)_{\text{good}} \quad \operatorname{Re} \lambda \|W\|^2 + \int |\bar{u}'| |W|^2 \leq 0,$$

whence $W \equiv 0$, a contradiction. ■

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