Galloping Instability of Detonation Waves

Benjamin Texier and Kevin Zumbrun

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The reacting Navier-Stokes equations

$$
\text{(rNS)}\begin{cases}
\partial_t \tau - \partial_x u = 0 \\
\partial_t u + \partial_x p = \partial_x(\nu\tau^{-1}\partial_x u) \\
\partial_t E + \partial_x(pu) = \partial_x\left(\kappa\tau^{-1}\partial_x T + \nu\tau^{-1}u\partial_x u + \boxed{qd\partial_x z}\right) \\
\boxed{\partial_t z + k\phi(T)z} = \boxed{\partial_x(d\partial_x z)}
\end{cases}
$$

$t \geq 0$, $\underline{x \in \mathbb{R}}$

$\tau > 0$ specific volume, u velocity, $E > 0$ total specific energy

$0 \leq z \leq 1$ mass fraction of the reactant

$p = \Gamma\tau^{-1}e,\ T = c^{-1}e$

e internal energy: $E = e + \frac{u^2}{2} + qz$

ϕ ignition function: $\phi \equiv 0$ for $T \leq T_i$, $\phi > 0$ for $T > T_i$.

The reacting Navier-Stokes equations

After the change of reference frame: $x \to x - s(\varepsilon)t$,

$$\text{(rNS)} \qquad \partial_t U + \partial_x F(\varepsilon, U) = \partial_x(B(\varepsilon, U)\partial_x U) + G(\varepsilon, U).$$

$$U = (\tau, u, E, z),$$
$$\varepsilon := (q, \nu, \kappa, d, k) \in \mathbb{R} \times \mathbb{R}_+^4.$$

Strong detonations

A right-going family of *viscous strong detonations* is a family $\{\bar{U}^\varepsilon\}_\varepsilon$ of standing-wave solutions

$$\bar{U}^\varepsilon(x,t) = \bar{U}^\varepsilon(x), \qquad \lim_{x\to\pm\infty} \bar{U}^\varepsilon(x) = U^\varepsilon_\pm,$$

of (rNS) associated with speeds $s(\varepsilon)$ satisfying

$$s(\varepsilon) > 0, \qquad \text{uniformly in } \varepsilon,$$

connecting a burned state on the left to an unburned state on the right,

$$z^\varepsilon_- \equiv 0, \quad z^\varepsilon_+ \equiv 1,$$

and satisfying the Lax characteristic conditions

$$\sigma_- := \sigma(U^\varepsilon_-) > s(\varepsilon) > \sigma_+ := \sigma(U^\varepsilon_+),$$

uniformly in ε, where $\sigma := (p\partial_e p - \partial_\tau p)^{1/2}$ sound speed.

Exponential convergence to the endstates

Lemma

A family $\bar{U}^\varepsilon$ of viscous strong detonation solution of (rNS), *under above Assumptions, satisfies*

$$|\partial_x^k(\bar{U}^\varepsilon - U_\pm^\varepsilon)(x)| \le Ce^{-\eta|x|}, \qquad x \gtrless 0,$$

uniformly in ε, *for some* $C > 0$, $\eta > 0$.

Galloping waves

Experimental observation

If the reaction heat is large enough, detonation waves exhibit "galloping" instability behavior, i.e., time-periodic variations of velocity and profile.

Analysis

Galloping waves are evidenced by perturbation analysis around a background family of strong detonations, under an appropriate spectral assumption.

The linearized equations

Reacting Navier-Stokes

$$(\text{rNS}) \qquad \partial_t U = \mathcal{F}(\varepsilon, U)$$

The linearized equations

Reacting Navier-Stokes

$$\text{(rNS)} \qquad \partial_t U = -\partial_x(F(\varepsilon, U)) + \partial_x(B(\varepsilon, U)\partial_x U) + G(\varepsilon, U).$$

Linearized (rNS) about $\bar{U}^\varepsilon$:

$$\begin{aligned} \partial_t U &= L(\varepsilon)U \\ L(\varepsilon)V &:= \partial_x\big(-\partial_U F(\varepsilon, \bar{U}^\varepsilon)V + (\partial_U B(\varepsilon, \bar{U}^\varepsilon)\, V)\partial_x \bar{U}^\varepsilon + B(\varepsilon, \bar{U}^\varepsilon)\partial_x V\big) \\ &\quad + \partial_U G(\varepsilon, \bar{U}^\varepsilon)V. \end{aligned}$$

Spectrum of L

Observation $\sharp 1$: $0 \in \sigma_p(L(\varepsilon))$.

Obervation $\sharp 2$: [Henry] The essential spectrum is confined to the complement of

$$\{\lambda \in \mathbb{C}, \quad \Re\lambda > -\eta \frac{|\Im\lambda|}{1+|\Im\lambda|}\}, \qquad \text{for some } \eta > 0.$$

Observation $\sharp 3$: [Lyng and Zumbrun] The Evans function vanishes at order one at the origin:

$$D(\varepsilon, 0) = 0, \qquad D'(\varepsilon, 0) \neq 0.$$

Assumption: Poincaré-Hopf bifurcation

Assumption

In a neighborhood of $\{\Re\lambda \geq 0\} \setminus \{0\}$, *the only zeros of* D, *besides the simple root at the origin, are a pair* $\lambda_\pm$ *of complex conjugate eigenvalues of* $L(\varepsilon)$,

$$\lambda_\pm(\varepsilon) = \gamma(\varepsilon) \pm i\tau(\varepsilon),$$

satisfying

$$\gamma(0) = 0, \qquad \tau(0) \neq 0, \qquad \gamma'(0) \neq 0.$$

Poincaré-Hopf theorem in $\mathbb{R}^2$

$$y' = \begin{pmatrix} \gamma(\varepsilon) & -\tau(\varepsilon) \\ \tau(\varepsilon) & \gamma(\varepsilon) \end{pmatrix} y' + G(\varepsilon, y).$$

Assume

$$G(\varepsilon, y) = O(|y|^2), \qquad \gamma(0) = 0, \qquad \tau(0) \neq 0, \qquad \gamma'(0) \neq 0.$$

Theorem

For any sufficiently small $a > 0$, there exists a unique nontrivial periodic orbit y_a of the ODE with initial radius a, $|y_a(0)| = a$, and parameter value $\varepsilon(a)$.

A bifurcation theorem

Theorem

Consider (rNS) *and a family of strong detonations* $\bar{U}^\varepsilon$, *under the ideal-gas assumption and the Poincaré-Hopf assumption.*
For $r \geq 0$, $\eta > 0$ *sufficiently small and* $C > 0$ *sufficiently large, there are* C^1 *functions* $r \to \varepsilon(r)$, $r \to T(r)$,

$$\varepsilon(0) = 0, \qquad T(0) = 2\pi/\tau(0),$$

and a family of time-periodic solutions $\mathbf{U}^r(x,t)$ *of* (rNS) *with* $\varepsilon = \varepsilon(r)$, *of period* $T(r)$, *with*

$$C^{-1} r \leq \|\mathbf{U}^r - \bar{U}^{\varepsilon(r)}\|_{H^2_\eta} \leq Cr.$$

Sketch of proof

From Reacting Navier-Stokes equations

$$(\text{rNS}) \qquad \partial_t U + \partial_x F(\varepsilon, U) = \partial_x (B(\varepsilon, U)\partial_x U) + G(\varepsilon, U)$$

To hyperbolic-parabolic systems (e.g., Navier-Stokes)

$$(\text{NS}) \qquad \partial_t U + \partial_x F(\varepsilon, U) = \partial_x (\underbrace{B(\varepsilon, U)}_{\text{singular}} \partial_x U).$$

Sketch of proof

From Reacting Navier-Stokes equations

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To hyperbolic-parabolic systems (e.g., Navier-Stokes)

$$\text{(NS)} \qquad \begin{cases} \partial_t u_1 + \partial_x F_1(\varepsilon, u) = 0, \\ \partial_t u_2 + \partial_x F_2(\varepsilon, u) = \partial_x (\underbrace{b(\varepsilon, u)}_{\text{full rank}} \partial_x u_2). \end{cases}$$

Step 0: the Perturbation equations

Let

$$U = \bar{U}^{\varepsilon} + \dot{U}.$$

Then $\dot{U}$ solves

$$\partial_t \dot{U} + L(\varepsilon)\dot{U} = \partial_x Q(\varepsilon, \bar{U}^{\varepsilon}, \dot{U}). \tag{1}$$

Step 1: Coordinatization

Let $\phi_{\pm}^{\varepsilon}$ be the bifurcation eigenfunctions of $L(\varepsilon)$, $\tilde{\phi}_{\pm}^{\varepsilon}$ the bifurcation eigenfunctions of $L(\varepsilon)^*$.
Let Π be the projector over span $\{\phi_{+}^{\varepsilon}, \phi_{-}^{\varepsilon}\}$, such that

$$\Pi f = (\tilde{\phi}_{-}^{\varepsilon}, f)_{L^2}\, \phi_{-}^{\varepsilon} + (\tilde{\phi}_{+}^{\varepsilon}, f)_{L^2}\, \phi_{+}^{\varepsilon}.$$

Let

$$\Pi \dot{U} = w_1 \Re \phi_{+}^{\varepsilon} + w_2 \Im \phi_{+}^{\varepsilon}, \qquad (1 - \Pi)\dot{U} = v.$$

Then, (w, v) solves

$$\begin{cases} \partial_t w = \begin{pmatrix} \gamma(\varepsilon) & -\tau(\varepsilon) \\ \tau(\varepsilon) & \gamma(\varepsilon) \end{pmatrix} w + \Pi \partial_x Q(\varepsilon, \bar{U}^{\varepsilon}, \dot{U}), \\ \partial_t v = (1 - \Pi) L(\varepsilon) v + (1 - \Pi)\partial_x Q(\varepsilon, \bar{U}^{\varepsilon}, \dot{U}). \end{cases} \tag{2}$$

Step 2: Poincaré return map

The solution (w, v) of (1) is T-time periodic if and only if

$$(w, v)(T) = (w, v)(0) =: (a, b),$$

or

$$f(\varepsilon, T, a, b) = 0, \qquad g(\varepsilon, T, a, b) = 0,$$

with the notation

$$\begin{aligned}
f(\varepsilon, T, a, b) &:= (\mathrm{Id} - R(\varepsilon, T))a - N_1(\varepsilon, T, a, b),\\
g(\varepsilon, T, a, b) &:= (\mathrm{Id} - S(\varepsilon, T))b - N_2(\varepsilon, T, a, b),\\
R(\varepsilon, T) &:= e^{T(\gamma \mathrm{Id} + \tau J)},\\
S(\varepsilon, T) &:= e^{T(1-\Pi)L(\varepsilon)}\\
N_1(\varepsilon, T, a, b) &:= \int_0^T R(\varepsilon, T-t)\Pi\partial_x Q(\varepsilon, \bar{U}^\varepsilon, \dot{U})(t)\, dt,\\
N_2(\varepsilon, T, a, b) &:= \int_0^T S(\varepsilon, T-t)(1-\Pi)\partial_x Q(\varepsilon, \bar{U}^\varepsilon, \dot{U})(t)\, dt.
\end{aligned}$$

Step 3: Poincaré-Hopf bifurcation.

Suppose that we can solve $g = 0$ by

$$b = \beta(\varepsilon, T, a),$$

with

$$\|\beta, \partial_\varepsilon \beta\|_{H^2_\eta} \leq C|a|.$$

Then, solve

$$f_*(\varepsilon, T, a) = f(\varepsilon, T, a, \beta(\varepsilon, T, a)),$$

by standard Poincaré-Hopf bifurcation in $\mathbb{R}^2$.

Step 4: Solving $g = 0$.

We show that $g = 0$:

$$(\mathrm{Id} - S(\varepsilon, T))b - N_2(\varepsilon, T, a, b) = 0,$$

can be put in the form

$$b = \underbrace{(\mathrm{Id} - S(\varepsilon, T))^{-1}}_{\text{right inverse}} N_2(\varepsilon, T, a, b), \tag{3}$$

and a quadratic bound on N_2 gives a solution to (3) in H^2_η by the Banach fixed point theorem, for small a.

Convergence of the Neumann series for Id $-$ S

Lemma

The operator $Id - S(\varepsilon, T)$ *has a right inverse*

$$(Id - S(\varepsilon, T))^{-1}: \quad H^2_\eta \cap \partial_x(H^1_{\eta^2}) \to H^2,$$

that belongs to $\mathcal{L}(H^2_\eta \cap \partial_x(H^1_{\eta^2}), H^2_\eta)$, *locally uniformly in*

$$(\varepsilon, T) \in [-\varepsilon_0, \varepsilon_0] \times (0, +\infty).$$

Sketch of proof of Lemma

Green function representation of the semi-group:

$$S(\varepsilon, t)f = \big(G(\varepsilon, x, t; y), f(y)\big)_{\mathcal{D}', \mathcal{D}(\mathbb{R}_y)}.$$

Theorem (Mascia-Howard-Zumbrun)

$$G \simeq K + J,$$
$$K = t^{-1/2} e^{-(x-y-at)^2/Ct}$$
$$J = (\bar{U}^{\varepsilon})'(x) \int_{-\infty}^{y-at/C\sqrt{t}} e^{-z^2}\, dz.$$

To prove the Lemma, it is sufficient to prove the *long-time* estimates:

$$\Big\| \int_{-\infty}^{+\infty} \Big(\int_T^{\infty} K_y(x, t; y)\, dt \Big) f(y)\, dy \Big\|_{L^2} \leq C \|f\|_{L^1}$$
$$\Big\| \int_{-\infty}^{+\infty} \Big(\int_T^{\infty} J_y(x, t; y)\, dt \Big) f(y)\, dy \Big\|_{L^2} \leq C \|f\|_{L^1}.$$