

Existence and Stability of Viscous Shock Profiles for Isentropic MHD with Infinite Electrical Resistivity

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Overview

- Isentropic gas
- Parallel MHD
- MHD with infinite electrical resistivity
- Known instabilities in multi-D problem
- Phase space
- Stability setup
- Numerical approximation

Magnetohydrodynamics (MHD)

$$\left\{ \begin{array}{l} v_t - u_{1x} = 0, \\ u_{1t} + (p + (1/2\mu_0)(B_2^2 + B_3^2))_x = (((2\mu + \eta)/\nu)u_{1x})_x, \\ u_{2t} - ((1/\mu_0)IB_2)_x = ((\mu/\nu)u_{2x})_x, \\ u_{3t} - ((1/\mu_0)IB_3)_x = ((\mu/\nu)u_{3x})_x, \\ (\nu B_2)_t - (Iu_2)_x = ((1/\sigma\mu_0\nu)B_{2x})_x, \\ (\nu B_3)_t - (Iu_3)_x = ((1/\sigma\mu_0\nu)B_{3x})_x, \end{array} \right.$$

Hyperbolic wherever p is monotone.

Rescaled Equations

$$(v, u, \mu_0, x, t, B, a) \rightarrow \left(\frac{v}{\epsilon}, -\frac{u}{\epsilon S}, \epsilon \mu_0, -\epsilon S(x - st), \epsilon S^2 t, -\frac{B}{S}, \frac{a \epsilon^{-\gamma-1}}{S^2} \right)$$

$$\left\{ \begin{array}{l} v_t + v_x - u_{1x} = 0 \\ u_{1t} + u_{1x} + \left(a v^{-\gamma} + \left(\frac{1}{2\mu_0} \right) (B_2^2 + B_3^2) \right)_x = (2\mu + \eta) \left(\frac{u_{1x}}{v} \right)_x \\ u_{2t} + u_{2x} - \left(\frac{1}{\mu_0} I B_2 \right)_x = \mu \left(\frac{u_{2x}}{v} \right)_x \\ u_{3t} + u_{3x} - \left(\frac{1}{\mu_0} I B_3 \right)_x = \mu \left(\frac{u_{3x}}{v} \right)_x \\ (v B_2)_t + (v B_2)_x - (I u_2)_x = \left(\left(\frac{1}{\sigma \mu_0 v} \right) B_{2x} \right)_x \\ (v B_3)_t + (v B_3)_x - (I u_3)_x = \left(\left(\frac{1}{\sigma \mu_0 v} \right) B_{3x} \right)_x \end{array} \right.$$

Profile Equations

Traveling wave solution: $(v, u_1, u, B)(x, t) = (v, u_1, \tilde{u}, \tilde{B})(x - st)$

$$(v, u_1, \tilde{u}, \tilde{B})(\pm\infty) = (v, u_1, \tilde{u}_2, \tilde{B})_{\pm}.$$

$$\begin{cases} v' - u_1' = 0, \\ u_1' + (p + (1/2\mu_0)|\tilde{B}|^2)' = (((2\mu + \eta)/v)u_1')' \\ \tilde{u}' - ((1/\mu_0)I\tilde{B})' = ((\mu/v)\tilde{u}')' \\ (v\tilde{B})' - (I\tilde{u})' = ((1/\sigma\mu_0v)\tilde{B}')' \end{cases}$$

Profile Equations

$$u = u_1, w := \tilde{u}, B = \tilde{B}$$

$$(2\mu + \eta)v' = v(v - 1) + v(p - p_-) + \frac{v}{2\mu_0}(B^2 - B_-^2),$$

$$\mu w' = vw - \frac{vl}{\mu_0}(B - B_-),$$

$$\frac{1}{\sigma\mu_0}B' = v^2B - vB_- - lvw,$$

with $u \equiv v - 1$.

Profile Equations, $\sigma = \infty$

$$\sigma = \infty$$

$$(v\tilde{B})' - (I\tilde{u})' = 0,$$

$$(vB)' - (Iw)' = 0$$

$$B = \frac{B_- + Iw}{v}.$$

$$(2\mu + \eta)v' = v(v - 1) + v(p - p_-) + \frac{1}{2\mu_0 v}((B_- + Iw)^2 - v^2 B_-^2),$$

$$\mu w' = vw - \frac{I}{\mu_0}(B_-(1 - v) + Iw).$$

RH conditions

$$u = u_1, \quad B = (B_2, B_3), \quad w = (u_2, u_3),$$

$$-s[v] = [u],$$

$$-s[u] = - \left[p + \frac{B^2}{2\mu_0} \right],$$

$$-s[w] = I \left[\frac{B}{\mu_0} \right],$$

$$-s[vB] = I[w].$$

Important parameters

$$J := \frac{(B_{2-})^2}{2\mu_0} \text{ and } K := \frac{I^2}{\mu_0}.$$

(Note that, under the rescaling that we used, $I = -\frac{I}{s}$,

$J = \frac{B_{2-}^2}{2\epsilon s^2 \mu_0} = \frac{B_{2-}^2}{2v_- s^2 \mu_0}$, $K = \frac{(I)^2}{\epsilon s^2 \mu_0} = \frac{(I)^2}{v_- s^2 \mu_0}$ in the original coordinates.)

Unique RH solution

$$u_+ = v_+ - 1, \quad B_{2-} = \left(\frac{v_+ - K}{1 - K}\right) B_{2+}, \quad w_+ = \frac{K}{l} \left(\frac{1 - v_+}{1 - K}\right) B_{2+},$$
$$a = \left(\frac{1 - v_+}{v_+^{-\gamma} - 1}\right) \left(1 - \frac{B_{2+}^2}{2\mu_0} \frac{(1 + v_+ - 2K)}{(1 - K)^2}\right) = \left(\frac{1 - v_+}{v_+^{-\gamma} - 1}\right) \left(1 - J \frac{(1 + v_+ - 2K)}{(v_+ - K)^2}\right).$$

This is physically meaningful if and only if $a > 0$, or

$$-1 < v_+ - 1 < 2(K - 1) + (1 - K)^2 \left(\frac{2\mu_0}{B_{2+}^2}\right).$$

(For $K \geq 1/2$, this gives no restriction. For $K < 1/2$, $B_{2+}^2 < \frac{2\mu_0(1-K)^2}{1-2K}$ or $J < \frac{(v_+ - K)^2}{1-2K}$.)

Shock type

Consider a general system of conservation laws

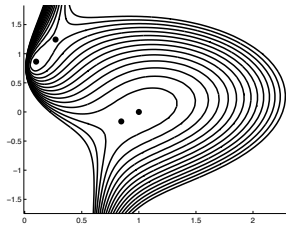
$$U_t + F(U)_x = (\mathcal{B}(U)U_x)_x, \quad U \in \mathbb{R}^n$$

Inviscid shock waves correspond to triples (U_-, U_+, s) satisfying the Rankine–Hugoniot conditions

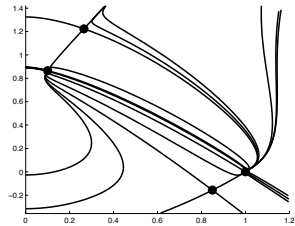
$$[F(U)] - s[U] = 0,$$

where $[h] := h(U_+) - h(U_-)$ denotes the jump in quantity h across the shock. The *type* of the shock wave is defined by the degree of compressivity

$$\ell := \dim \mathcal{U}(dF(U_-) - sl) + \dim \mathcal{S}(dF(U_+) - sl) - n,$$



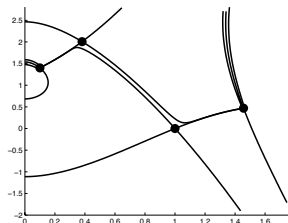
(a)



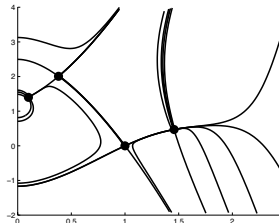
(b)

Figure: Typical phase portrait for MHD with two variables and infinite electric resistivity ($\sigma = \infty$). Parameter values are $\gamma = 2$, $\tau_+ = 0.1$, $I = 0.7$, $B_{2+} = 0.7$, and $\mu_0 = 1$. In Figure (a) we plot level sets of ϕ and in Figure (b) we draw the phase portrait.

Phase portraits: MHD



(a)



(b)

Figure: Transition to undercompressive profile. keeping $\tau = 2\mu + \eta = 1$ and letting $\mu \rightarrow 0$, we find that the overcompressive family is squeezed to an undercompressive connection somewhere between $\mu = 0.185$ (Figure (a)) and $\mu = 0.17$ (Figure (b)).

Spectral Stability

Linearization about the profile

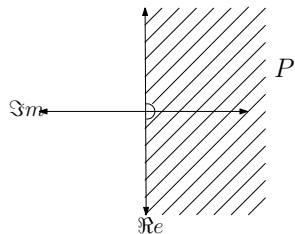
$$v_t = \underbrace{(d\mathcal{F}(\hat{u}) - s\partial_x)}_{L_v} v + \underbrace{Q(v, v_x, v_{xx}, \dots)}_{\text{Higher order}}.$$

The eigenvalue problem is a BVP.

$$L_v = \lambda v, \quad v^{(k)}(\pm\infty) = 0.$$

Spectral Stability when $\sigma(L) \cap P = \emptyset$,

where $P = \{\lambda \mid \Re(\lambda) \geq 0\} \setminus \{0\}$.



Artificial Viscosity

Existence and Asymptotic Behavior

Madja and Pego (83)

Small-Amplitude Spectral Stability

Goodman (86)

Nonlinear Stability

T.P. Liu (86), partial result
Szepessy and Xin (92)

Pointwise Greens Function Bounds

T.P. Liu (97)

Spectral Stability \Rightarrow Nonlinear Stability

Zumbrun and Howard (98,00)

Real Viscosity

Existence and Asymptotic Behavior

Pego (82)

Dissipativity Condition

Kawashima (83)

Small-Amplitude Spectral Stability (Navier Stokes)

Matsumura and Nishihara (Isentropic)(84)
Kawashima, Matsumura and Nishihara (86)

Small-Amplitude Spectral Stability

Humpherys and Zumbrun (02)

Spectral Stability \Rightarrow Nonlinear Stability

Mascia and Zumbrun (01,04,06)

Eigenvalue Problem

Write the eigenvalue problem

$$\lambda v = Lv, \quad -\infty < x < \infty,$$

as a first-order system

$$\begin{cases} W' = A(x, \lambda)W, & W \in \mathbb{C}^n \\ W(\pm\infty) = 0. \end{cases}$$

- Assume $A(x, \lambda)$ is consistently split in x .
- Assume $A(x, \lambda)$ is asymptotically constant in x , that is, $\lim_{x \rightarrow \pm\infty} A(x, \lambda) = A_{\pm}(\lambda)$.

Evans Function

Define at $x = 0$

$$D(\lambda) = \underbrace{W_1^+ \wedge \dots \wedge W_k^+}_{S^+(\lambda)} \wedge \underbrace{W_{k+1}^- \wedge \dots \wedge W_n^-}_{U^-(\lambda)}$$

where $\{W_i^+\}_{i=1}^k$ and $\{W_j^-\}_{j=k+1}^n$ are analytic bases of the stable/unstable manifolds of $x = \pm\infty$, respectively.

Theorem (Alexander, Gardner, Jones)

The Evans Function enjoys the following properties:

- $D(\lambda)$ is analytic to the right of the essential spectrum.
- Degree of roots of $D(\lambda)$ match multiplicity of $\sigma_P(L)$.

Remark

Since $D(\lambda)$ is analytic in right half plane we can use winding number computations to test for spectral stability.

Linearized equations

$(\hat{v}, \hat{u}_1, \hat{u}_2, \hat{B}_2)$

$$v_t + v_x - u_{1x} = 0$$

$$u_{1t} + u_{1x} + (-a\gamma\hat{v}^{-\gamma-1}v + (1/\mu_0)(\hat{B}_2 B_2))_x = \tau(u_{1x}/\hat{v} - \hat{u}_{1x}v/\hat{v}^2)_x$$

$$u_{2t} + u_{2x} - (I/\mu_0)B_{2x} = \mu(u_{2x}/\hat{v} - \hat{u}_{2x}v/\hat{v}^2)_x$$

$$\tilde{\alpha}_t + \tilde{\alpha}_x - Iu_{2x} = (\sigma\mu_0)^{-1}(B_{2x}/\hat{v} - \hat{B}_{2x}v/\hat{v}^2)_x,$$

where $\tilde{\alpha} = \hat{v}B_2 + v\hat{B}_2$, so that $B_2 = (\tilde{\alpha} - \hat{B}_2 v)/\hat{v}$.

Eigenvalue problem

$$\lambda v + v' - u_1' = 0$$

$$\lambda u_1 + u_1' - (h(\hat{v})v/\hat{v}^{\gamma+1})' = -(\hat{B}_2(\bar{\alpha} - \hat{B}_2 v)/(\mu_0 \hat{v}))' + \tau(u_1'/\hat{v})'$$

$$\lambda u_2 + u_2' - (I/\mu_0)(\bar{\alpha}/\hat{v} - \hat{B}_2 v/\hat{v})' = \mu(u_2'/\hat{v} - \hat{u}_2' v/\hat{v}^2)'$$

$$\lambda \bar{\alpha} + \bar{\alpha}' - I u_2' = (\sigma \mu_0)^{-1} (\hat{v}^{-1} (\bar{\alpha}/\hat{v} - \hat{B}_2 v/\hat{v})' - \hat{B}_2' v/\hat{v}^2)',$$

$$h(\hat{v}) = -\hat{v}^{\gamma+1} (\tau \hat{u}_1'/\hat{v}^2 - a \gamma \hat{v}^{-\gamma-1})$$

$$= -\hat{v}^{\gamma+1} (\tau \hat{v}'/\hat{v}^2 - a \gamma \hat{v}^{-\gamma-1})$$

$$= -\hat{v}^{\gamma+1} (\hat{v}^{-2} (\hat{v}(\hat{v} - 1) + a \hat{v}^{1-\gamma} - a \hat{v} + (2\mu_0 \hat{v})^{-1} ((B_{2-} + I \hat{u}_2)^2 - \hat{v}^2 B_{2-}^2)) - a \gamma \hat{v}^{-\gamma-1}).$$

Integrated coordinates

$$u(x) = \int_{-\infty}^x u_1(z) dz, \quad w = \int_{-\infty}^x u_2(z) dz,$$

$$V = \int_{-\infty}^x v(z) dz, \quad \text{and } \alpha = \int_{-\infty}^x \tilde{\alpha}(z) dz$$

$$\lambda V' + V'' - u'' = 0$$

$$\lambda u' + u'' - (h(\hat{v})V'/\hat{v}^{\gamma+1})' = -(\hat{B}_2(\alpha' - \hat{B}_2 V')/(\mu_0 \hat{v}))' + \tau(u''/\hat{v})'$$

$$\lambda w' + w'' - (I/\mu_0)(\alpha'/\hat{v} - \hat{B}_2 V'/\hat{v})' = \mu(w''/\hat{v} - \hat{u}'_2 V'/\hat{v}^2)'$$

$$\lambda \alpha' + \alpha'' - I w'' = (\sigma \mu_0)^{-1} (\hat{v}^{-1} ((\alpha' - \hat{B}_2 V')/\hat{v})' - \hat{B}'_2 V'/\hat{v}^2)'$$

Integrating from $-\infty$ to x we obtain

$$\begin{aligned}\lambda V + V' - u' &= 0 \\ \lambda u + u' - h(\hat{v})V'/\hat{v}^{\gamma+1} &= -\hat{B}_2(\alpha' - \hat{B}_2 V')/(\mu_0 \hat{v}) + \tau(u''/\hat{v}) \\ \lambda w + w' - (I/\mu_0)(\alpha'/\hat{v} - \hat{B}_2 V'/\hat{v}) &= \mu(w''/\hat{v} - \hat{u}'_2 V'/\hat{v}^2) \\ \lambda \alpha + \alpha' - Iw' &= (\sigma\mu_0)^{-1}(\hat{v}^{-1}(\alpha'/\hat{v} - \hat{B}_2 V'/\hat{v})' - \hat{B}'_2 V'/\hat{v}^2).\end{aligned}$$

Eigenvalue problem, $\sigma = \infty$

$$u_3 \equiv B_3 \equiv 0, \sigma = \infty,$$

$$\lambda V + V' - u' = 0$$

$$\lambda u + u' - h(\hat{v})V'/\hat{v}^{\gamma+1} = -\hat{B}_2(\alpha' - \hat{B}_2 V')/(\mu_0 \hat{v}) + \tau(u''/\hat{v})$$

$$\lambda w + w' - (I/\mu_0)(\alpha'/\hat{v} - \hat{B}_2 V'/\hat{v}) = \mu(w''/\hat{v} - \hat{u}'_2 V'/\hat{v}^2)$$

$$\lambda \alpha + \alpha' - Iw' = 0.$$

Eigenvalue problem, $\sigma = \infty$

$$(u, v, v', w, \mu w', \alpha)^T, u'' = \lambda V' + V'', K = I^2/\mu_0$$

$$u' = \lambda V + V'$$

$$V'' = \frac{\lambda \hat{v} V}{\tau} + \left(-\frac{h(\hat{v})}{\tau \hat{v}^\gamma} - \lambda + \frac{\hat{v}}{\tau} - \frac{\hat{B}_2^2}{\mu_0 \tau} \right) V' + \frac{\lambda \hat{v} u}{\tau} + \frac{I \hat{B}_2 w'}{\mu_0 \tau} - \frac{\lambda \hat{B}_2 \alpha}{\mu_0 \tau}$$

$$w'' = \left(\frac{\hat{u}'_2}{\hat{v}} + \frac{I \hat{B}_2}{\mu_0 \mu} \right) V' + \frac{\lambda \hat{v} w}{\mu} + \frac{\hat{v} w'}{\mu} - \frac{K w'}{\mu} + \frac{\lambda I \alpha}{\mu_0 \mu}$$

$$\alpha' = I w' - \lambda \alpha.$$

First order system, $\sigma = \infty$

$$W' = A(x, \lambda)W$$

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\lambda \hat{v}}{\tau} & \frac{\lambda \hat{v}}{\tau} & f(\hat{v}) - \lambda - \frac{\hat{B}_2^2}{\mu_0 \tau} & 0 & \frac{I \hat{B}_2}{\mu_0 \mu \tau} & -\frac{\lambda \hat{B}_2}{\mu_0 \tau} \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & \frac{\mu \hat{u}'_2}{\hat{v}} + \frac{I \hat{B}_2}{\mu_0} & \lambda \hat{v} & \frac{\hat{v} - K}{\mu} & \frac{\lambda I}{\mu_0} \\ 0 & 0 & 0 & 0 & \frac{I}{\mu} & -\lambda \end{pmatrix},$$

$$W = (u, V, V', w, \mu w', \alpha)^T, \quad f(\hat{v}) = \tau^{-1}(\hat{v} - \hat{v}^{-\gamma} h(\hat{v})).$$

Numerical stability investigation: profile approximation

- STABLAB (quite general)
- bvp4c, bvp5c, bvp6c (Lobatto quadrature scheme)
- finite computational domain $[-L, L]$
- L_{\pm} chosen experimentally, $|U(\pm L_{\pm}) - U_{\pm}| < \text{error}$
- projective boundary conditions $M_{\pm}(U - U_{\pm}) = 0$

Approximation of the Evans function

Polar coordinate method (Humpherys-Zumbrun)

$$\mathcal{W} = r \Omega,$$

$$\mathcal{W} = W_1 \wedge \cdots \wedge W_k$$

$$\Omega = \omega_1 \wedge \cdots \wedge \omega_k$$

$$D(\lambda) = \mathcal{W}^- \wedge \mathcal{W}^+|_{x=0} = \det(W_1^-, \dots, W_k^-, W_{k+1}^+, \dots, W_N^+)|_{x=0}.$$

$$\mathcal{W}^-(-L_-) \sim e^{-\mu L_-} (R_1^- \wedge \cdots \wedge R_k^-)$$

$$\mu = \text{Trace} A_-|_{U(A_-)}$$

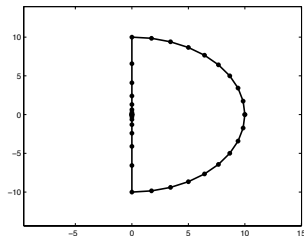
Shooting and initialization

- ode45 (adaptive 4th-order Runge-Kutta-Fehlberg)
- Error tolerance: AbsTol=1e-8, RelTol=1e-6
- Kato's ODE
- Second-order algorithm (Zumbrun)

Winding number computation

$$S := \partial(B(0, R) \cap \{\Re \lambda \geq 0\})$$

- 20 points taken quadratic in modulus
- winding number calculation
- require relative error be less than 0.2
- Rouché's Theorem



HF convergence

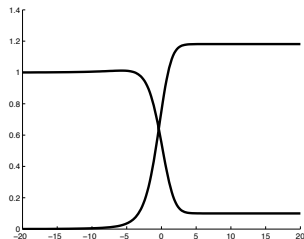
Proposition

$$\lim_{|\lambda| \rightarrow \infty} \frac{\tilde{D}(\lambda)}{e^{\alpha\lambda^{1/2}}} = C \text{ uniformly on } \Re\lambda \geq 0,$$

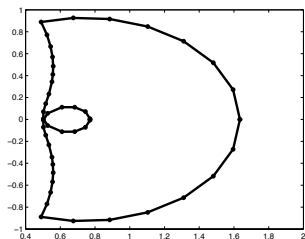
where α and C are constants.

$$\log \tilde{D}(\lambda) = \log C + \alpha\lambda^{1/2} \quad |\lambda| \gg 1.$$

Transverse Evans function



(a)



(b)

Figure: Typical transverse Evans function output, parameter values $\gamma = 5/3$, $I = 0.6$, $B_+ = 1.4$, and $\mu_0 = 1$. In Figure (a) we display the nonmonotone profile. In Figure (b) we display the winding number computation.

Large Amplitude limit

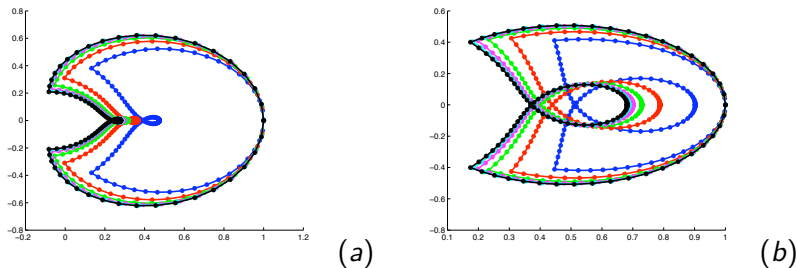


Figure: Large-amplitude limits, parameters $K = 2$, $J = 1$, $\gamma = 5/3$. In Figure (a), we display the image of the semicircle under \tilde{D} for a Lax 2-shock in two-rest point configuration in the $a \rightarrow 0$ limit, $a = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$, where $a = 10^{-8}$ corresponds to Mach number $\approx 10,954$. Convergence of contours appears to occur at $a \sim 10^{-6}$, or Mach number $\approx 1,095$. In Figure (b), for the same sequence of a -values, we display the images under the transverse Evans function, again suggestive of convergence.

Large amplitude limits

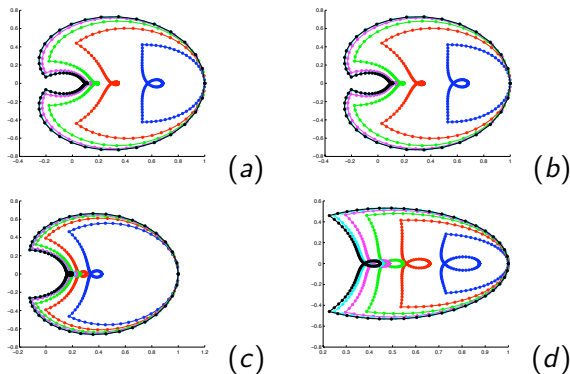


Figure: Large amplitude limits, parameters $K = 0.7$, $J = 0.5$, and $a = 10^{-1}, 10^{-2}, \dots, 10^{-k}$, getting smaller as necessary to see what appear to be convergence to a limit. (a). Lax 2-shock, v_2 to v_1 . (b). Intermediate Lax 2-shock, v_3 to v_1 . (c). overcompressive 1-2 shock, v_4 to v_1 . (d). Transverse Evans study for (c). In each case, we appear to obtain convergence at $a = 10^{-7}$, corresponding to Mach number $\approx 3,817$.

Experiments

$$(\gamma, K, J, v_+, \mu_0) \in \{7/5, 5/3\}$$

$$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 1.05, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}$$

$$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}$$

$$\times \{0.1\}$$

$$\times \{1.0\}.$$

$$(\gamma, v_+, I, B_{2+}, \mu_0) \in \{7/5, 5/3\}$$

$$\times \{0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 10^{-1}, 10^{-2}\}$$

$$\times \{0.2, 0.4, 0.6, 0.8, 1.2, 1.4, 1.6, 1.8, 2.0\}$$

$$\times \{0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0\}$$

$$\times \{1.0\}.$$

Typical computation time around 1 minute

Experiments: overcompressive family

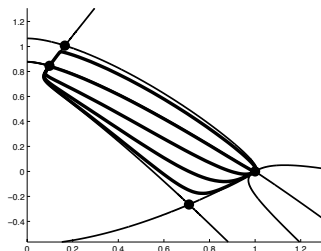
$(\gamma, K, J, a, \mu_0) \in \{7/5, 5/3\}$

$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95\}$

$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}$

$\times \{a_1, a_2, a_3, a_4, a_5\}$

$\times \{1.0\}$.



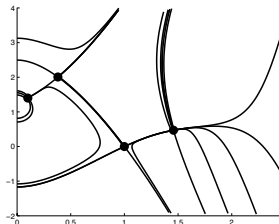
Experiments: undercompressive

$$(\gamma, \nu_+, K, J) \in \{7/5, 5/3\}$$

$$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$$

$$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$$

$$\times \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$$



The take away

- The system appears stable (known instabilities for multi-D model)
- Organizing the phase space important (knowing way around)
- adaptive mesh
- parallel again in future

Interesting directions

- Finite electrical resistivity
- Three dimensional profiles
- Full nonisentropic case
- Multidimensional stability (in x)