Notes on Constant Coefficients Hyperbolic Initial Boundary Value Problems

Guy Métivier *

March 16, 2015

These notes were first written for a lecture given at IHP Paris, on march 10, 2015, within the program "Stability of viscous shock waves and beyond" leaded by Kevin Zumbrun (Indiana University), 2014 Research Chair of the FSMP. They should evolve to include more materials, details, and proofs. Comments and remarks are welcome

1 The Cauchy problem

1.1 Introduction

Consider a constant coefficient system

$$L(\partial_t, \partial_x) = A_0\partial_t + \sum_{j=1}^{d} A_j \partial_{x_j} + B \tag{1.1}$$

and the Cauchy problem

$$\begin{cases} Lu = f, & t > 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.2}$$

We assume that A_0 is invertible, and multiplying the equation by A_0^{-1} we assume that $A_0 = \mathrm{Id}$.

Objectives :

- Introduce the notion of hyperbolicity

*Université de Bordeaux - CNRS, Institut de Mathématiques de Bordeaux, 351 Cours de la Libération , 33405 Talence Cedex, France

- Symmetrizers
- Well posed-ness of the Cauchy problem
- Finite speed of propagation

1.2 Analysis by Fourier synthesis

We look for solutions of the Cauchy problem in the class of temperate distributions in x, using the spatial Fourier transform

$$\hat{u}(\xi) = \mathscr{F}u(\xi) = \int e^{-\xi\cdot x} u(x)dx \tag{1.3}$$

the equation to solve is

$$\begin{cases} \partial_t \hat{u} + iA(\xi)\hat{u} = \hat{f}, & t > 0, \\ \hat{u}_{|t=0} = \hat{u}_0, \end{cases} \tag{1.4}$$

where

$$A(\xi) = \sum_{j=1}^{d} \xi_j A_j - iB. \tag{1.5}$$

Thus, assuming integrability in time for $\hat{f}$,

$$\hat{u}(t,\xi) = e^{-itA(\xi)}\hat{u}_0(\xi) + \int_0^t e^{i(s-t)A(\xi)} \hat{f}(s,\xi)ds. \tag{1.6}$$

In particular, for $f = 0$ this means that

$$\hat{u}(t,\xi) = e^{-itA(\xi)}\hat{u}_0(\xi). \tag{1.7}$$

This method is successful if one can perform the inverse Fourier transform, that is if the mutliplicator $e^{-itA(\xi)}$ acts in $\mathscr{S}'(\mathbb{R}^d)$.

A favorable case is when there are constant C, m and γ such that

$$\forall t \geq 0,\ \forall \xi \in \mathbb{R}^d, \quad \left| e^{-itA(\xi)} \right| \leq C\langle\xi\rangle^m \, e^{\gamma t} \tag{1.8}$$

in which case one can solve the Cauchy problem in $\mathscr{S}'$.

Lemma 1.1. *The estimate* (1.8) *for* $t = 1$ *implies that there is a constant* γ_1 *such that for all* $\xi \in \mathbb{R}^d$ *the eigenvalues of* $A(\xi)$ *satisfy* $\operatorname{Im} \lambda \leq \gamma_1$.

Proof. The estimate implies that the eigenvalues satisfy (with a new constant C)

$$e^{t\mathrm{Im}\,\lambda} = |e^{-it\lambda}| \le C\langle\xi\rangle^m. \tag{1.9}$$

Let $\mu(\sigma) = \sup(\mathrm{Im}\,\lambda)$ where the supremum is taken for the $(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^d$ such that $\det(A(\xi) - \lambda\mathrm{Id} = 0$, $\mathrm{Im}\,\lambda > 0$ and $|\lambda|^2 + |\xi|^2 = \sigma$. The estimate above implies that $\mu(\sigma)$ grows at most logarithmically as $\sigma \to \infty$. By a lemma on sub-algebraic functions (see e.g. Corollary A.2.6 in [Hör]), this implies that μ is bounded and the lemma follows. □

This is one way to motivate the following definition.

Definition 1.2. *The system is said to be hyperbolic in the time direction, if there is a constant* γ_1 *such that*

$$\det(\tau + A(\xi)) = 0 \qquad \Rightarrow \qquad |\mathrm{Im}\,\tau| \le \gamma_1. \tag{1.10}$$

Remark 1.3. The natural condition from the previous lemma is that the roots are located in $\mathrm{Im}\,\tau \ge -\gamma_1$. But, a consequence of Proposition 1.14 is that this property is preserved by reversing the time, and therefore the roots also satisfy $\mathrm{Im}\,\tau \le \gamma_2$. This is why we go directly to condition (1.10).

Proposition 1.4. *The system is hyperbolic in time if and only if he estimate* (1.8) *is satisfied.*

Proof. We have already said that the condition is necessary. We prove that it is sufficient. We use the representation

$$e^{-itA} = \frac{1}{2i\pi}\int_{\mathcal{C}} e^{-it\lambda}\big(\lambda\mathrm{Id} - A\big)^{-1} d\lambda \tag{1.11}$$

where $\mathcal{C}$ is a contour in $\mathbb{C}$ surrounding the spectrum of A. We choose $\mathcal{C}$ to be the union of the segment $\mathcal{C}_1 = \{|\mathrm{Re}\,\lambda| \le R\ ,\ \mathrm{Im}\,\lambda = -2\gamma_1\}$ and of the half circle $\mathcal{C}_2 = \{|\lambda + 2i\gamma_1| = R, \mathrm{Im}\,\lambda \ge -2\gamma_1\}$, and we choose $R = C\langle\xi\rangle$ with C large enough so that

$$\lambda \in \mathcal{C}_2 \quad \Rightarrow \quad \big|(\lambda\mathrm{Id} - A)^{-1}\big| \le C_1/R.$$

We claim that there is another constant C_2 such that

$$\lambda \in \mathcal{C}_1 \quad \Rightarrow \quad \big|(\lambda\mathrm{Id} - A)^{-1}\big| \le C_2(\gamma_1 + \langle\xi\rangle)^{N-1}\gamma_1^{-N}. \tag{1.12}$$

This implies the estimates (1.8) with $m = N$. Indeed, on $\mathcal{C}_1$, because the co-factors of the matrix $\lambda\mathrm{Id} - A$ are $O(\langle(\lambda,\xi)\rangle^{N-1}) = O(\langle\xi\rangle^{N-1})$, it is sufficient to prove

$$\big|\det(\lambda\mathrm{Id} - A)\big| \geq \gamma_1^N$$

But this is clear since

$$\det(\lambda\mathrm{Id} - A) = \prod(\lambda - \lambda_j)$$

and $|\lambda - \lambda_j| \geq |\mathrm{Im}\,\lambda - \mathrm{Im}\,\lambda_j| \geq \gamma_1$ on $\mathcal{C}_1$. □

The estimate (1.8) allows us to apply the inverse Fourier transform to (1.6) when the data are temperate in x. For instance, in the scale of Sobolev spaces, one can state:

Theorem 1.5. *If the system is hyperbolic in time, then the Cauchy problem is well posed in Sobolev spaces in the sense that there is a constant C such that for all $T > 0$, $\sigma \in \mathbb{R}$, for all $u_0 \in H^\sigma$ and $f \in L^1([0,T], H^\sigma)$ the Cauchy problem* (1.2) *has a unique solution $u \in C^0([0,T]; H^{\sigma-m})$ and*

$$\big\|u(t)\big\|_{H^{\sigma-m}} \leq Ce^{\gamma t}\big\|u_0\big\|_{H^\sigma} + C\int_9^t e^{\gamma(t-s)}\big\|f(s)\big\|_{H^\sigma} ds. \tag{1.13}$$

We have show that one can solve the Cauchy problem in Sobolev spaces. The formula above contains another information.

Proposition 1.6. *If the system is hyperbolic, there is a unique fundamental solution $E \in C^0(\mathbb{R}; H^{-\sigma})$ where $\sigma > N + \frac{1}{2}d$, of $LE = \delta\mathrm{Id}$ with $E = 0$ when $t < 0$.*

Proof. Let $\hat{U}(t,\xi)$ be the matrix valued function defined by (1.11). It is smooth in t and satisfies

$$\partial_t U + iA(\xi)U = 0, \qquad U(0,\xi) = \mathrm{Id}.$$

Let

$$\hat{E}(t,\xi) = 1_{\{t>0\}}U(t,\xi). \tag{1.14}$$

Then

$$\partial_t\hat{E} + iA(\xi)\hat{E} = \delta_{t=0}\mathrm{Id}, \qquad \hat{E} = 0 \ for\ t < 0, \quad |\hat{E}(t,\xi)| \leq C\langle\xi\rangle^N.$$

The inverse spatial Fourier transform of $\hat{E}$ has the desired property.

Conversely, if $LE_1 = \delta_{t=0,x=0}$ and $E = 0$ for $t < 0$, Holmgren's uniqueness theorem implies that for $t \geq 0$, E_1 has compact support in x. Hence its spatial Fourier transform $\hat{E}_1$ satisifes

$$\partial_t \hat{E}_1 + iA(\xi)\hat{E}_1 = \delta_{t=0}\mathrm{Id}, \qquad \hat{E}_1 = 0 \ for \ t < 0.$$

Thus we are reduced to uniqueness for o.d.e.'s and $\hat{E}_1 = \hat{E}$. □

1.3 A particular case: strongly hyperbolic systems

The best estimate one can expect by the method above is when $m = 0$ in (1.8). In this case, for $\mathrm{Im}\,\tau < -\gamma$

$$(\tau \mathrm{Id} + A(\xi))^{-1} = i \int_0^\infty e^{-it(\tau\mathrm{Id} + A(\xi))} dt, \tag{1.15}$$

the integral being absolutely convergent, and

$$(\gamma - \mathrm{Im}\,\tau)\big|(\tau \mathrm{Id} + A(\xi))^{-1}\big| \leq C. \tag{1.16}$$

Applying this estimate for $(\lambda\tau, \lambda\xi)$ and letting λ tend to $+\infty$ implies, first for $\mathrm{Im}\,\tau < 0$, then by symmetry for $\mathrm{Im}\,\tau \neq 0$, that

$$|\mathrm{Im}\,\tau|\big|(\tau \mathrm{Id} + A_p(\xi))^{-1}\big| \leq C. \tag{1.17}$$

where $A_p(\xi) = \sum \xi_j A_j$ is the principal part of A. Conversely, (1.17) implies (1.16) (with another constant C) for all $A = A_p + B$ for $\gamma = 2C|B|$.

There are several equivalent formulations of this condition.

Theorem 1.7. *Consider the homogeneous case $A(\xi) = \sum \xi_j A_j$ and $L(\tau, \xi) = \tau \mathrm{Id} + A(\xi)$. The following conditions are equivalent.*

i) For all matrix B, $L(\tau, \xi) + B$ is hyperbolic.

ii) $\sup_\xi \big|e^{iA(\xi)}\big| < +\infty$

iii) for all ξ, the matrix $A(\xi)$ has only real eigenvalues and is diagonalizable; moreover the eigen-projectors are uniformly bounded for $\xi \in \mathbb{R}^d$.

iv) for all ξ, the matrix $A(\xi)$ has only real eigenvalues and

$$\sup_{\xi \in \mathbb{R}^d} \sup_{\mathrm{Im}\,\tau < 0} |\mathrm{Im}\,\tau| \, \big|L(\tau, \xi)^{-1}\big| < +\infty. \tag{1.18}$$

v) for all $\xi \in \mathbb{R}^d$, there is a matrix $S(\xi)$ such that

$$S(\xi) = S^*(\xi), \qquad S(\xi)A(\xi) = A^*(\xi)S^*(\xi); \tag{1.19}$$

Moreover S is definite positive and $S(\xi)$ and $S(\xi)^{-1}$ are uniformly bounded for $\xi \in \mathbb{R}^d$.

Proof. See [Me1]. □

Definition 1.8. *The system $L(\partial)$ is strongly hyperbolic if its principal part satisfies one of the equivalent condition above.*

In particular, strong hyperbolicity depends only on the principal part of L, which is not the case for general hyperbolicity.

Theorem 1.9. *If L is strongly hyperbolic, then the Cauchy problem is well posed in L^2 in the sense that there are constants C and γ such that for all $T > 0$, for all $u_0 \in L^2$ and $f \in L^1([0,T], L^2)$ the Cauchy problem (1.2) has a unique solution $u \in C^0([0,T]; L^2)$ and*

$$\big\|u(t)\big\|_{L^2} \le Ce^{\gamma t}\big\|u_0\big\|_{L^2} + C\int_0^t e^{\gamma(t-s)}\big\|f(s)\big\|_{L^2}ds. \tag{1.20}$$

Conversely, if the Cauchy problem is well posed in L^2 in the sense above, the system is strongly hyperbolic.

Proof. The sufficiency is a particular case with $m = 0$ of Theorem 1.5. Conversely, the estimate (1.20) with $u_0 = 0$, implies that for $u \in C_0^\infty(\mathbb{R}^{1+d})$ and $\lambda > \gamma$.

$$(\lambda - \gamma)\big\|e^{-\lambda t}u\big\|_{L^2} \le C\big\|e^{-\lambda t}Lu\big\|_{L^2} \tag{1.21}$$

Using this estimate for

$$u(t,x) = e^{i\rho(t\tau + x\cdot\xi)}\chi(\rho^{\frac{1}{2}}x)a$$

with $\lambda = -\rho \mathrm{Im}\,\tau$ and letting ρ tend to $+\infty$ implies that for $\mathrm{Im}\,\tau < 0$

$$|\mathrm{Im}\,\tau|\big|a\big| \le C\big|L_p(\tau,\xi)a\big|$$

where L_p denotes the principal part of L. This is condition *iv)* of Theorem 1.7. □

Introduce the "energy "

$$\mathcal{E}(u) = \int \big(S(\xi)\hat{u}(\xi)\,,\,\hat{u}(\xi)\big)d\xi. \tag{1.22}$$

where S is the symmetrizer in condition v) of Theorem (1.7). Then $\mathcal{E}(u) \approx \|u\|^2_{L^2}$. In the homogenous case, the solutions of $Lu = 0$ satisfy

$$\frac{d}{dt}\mathcal{E}(u(t)) = 0. \tag{1.23}$$

More generally, if u is smooth enough,

$$\frac{d}{dt}\mathcal{E}(u(t)) = 2\mathrm{Re}\,\tilde{\mathcal{E}}(Lu(t), u(t)) \tag{1.24}$$

where $\tilde{\mathcal{E}}$ is the hermitian symmetric form associated to $\mathcal{E}$. This form is definite positive, hence using Cauchy Schwarz inequality, one has

$$\mathcal{E}(u(t))^{\frac{1}{2}} \le \mathcal{E}(u(t))^{\frac{1}{2}} + \int_0^t \mathcal{E}(f(s))^{\frac{1}{2}} ds \tag{1.25}$$

which is more precise than and implies (1.20). Estimates for zero-th order perturbations $L(\partial) + B$ follow from Gronwall's lemma.

Example 1.10. Symmetric hyperbolic systems in the sense of Friedrichs. *An important class of strongly hyperbolic systems has been introduced by Friedrichs [Fr1, Fr2]. The condition is that the symmetrizer S can be chosen independent of ξ. In this case, S is a constant matrix, which satisfies:*

$$SA_0 = (SA_0)^* \gg 0, \qquad SA_j = (SA_j)^*, \quad j = 1, \ldots, d. \tag{1.26}$$

In this case, the energy can be defined on the x side :

$$\mathcal{E}(u) = \int \big(Su(x)\,,\, u(x)\big)dx. \tag{1.27}$$

Note also that for symmetric systems as above, the cone of hyperbolic directions is the set of $\nu \in \mathbb{R}^{1+d}$ such that $SL_0(\nu)$ is definite positive.

1.4 Necessary conditions for the well posedness

Hyperbolicity is necessary, not only for the global (in space) well posed-ness but also in a local theory. Set

$$L(\tau, \xi) = \tau A_0 + \sum \xi_j A_j - iB, \qquad p(\tau, \xi) = \det L(\tau, \xi). \tag{1.28}$$

Recall that we assume that A_0 is invertible. The principal symbol is $L_0 = \tau A_0 + \sum \xi_j A_j$ and we set

$$p_0(\tau, \xi) = \det L_0(\tau, \xi). \tag{1.29}$$

Let H denote the half space $\{t > 0\}$. A minimal form for the well posed-ness of the Cauchy problem is the condition that

$$(WP) \qquad \begin{cases} \textit{for all } f \in C_0^\infty(H), \textit{ the equation } Lu = f \textit{ has a unique} \\ \textit{solution } u \in \mathscr{D}'(\mathbb{R}^{1+d}) \textit{ with support contained in } H. \end{cases}$$

Lemma 1.11. *If the condition WP is satisfied, then for all $f \in C^\infty$ with support in H the equation $Lu = f$ has a unique solution $u \in C^\infty$ with support in H. Moreover, if $\underline{\tilde{x}}$ is a point in H, there are constants C, R and s such that for all $u \in C^\infty$ with support in H:*

$$|u(\underline{\tilde{x}})| \leq C \sup_{|\alpha|\leq s,\ |\tilde{x}|\leq R} |\partial^\alpha Lu(\tilde{x})|. \tag{1.30}$$

Proof. See Lemma 12.3.2 in Hörmander [Hör] and the estimate (12.3.3) which follows. □

Theorem 1.12. *Suppose that the estimate* (1.30) *is satisfied. Then, p is hyperbolic in the time direction, i.e. there is a number γ_0 such that*

$$p(\tau, \xi) \neq 0 \qquad \mathit{if} \quad \xi \in \mathbb{R}, \quad \tau \in \mathbb{C} \quad \mathit{and} \quad \operatorname{Im} \tau < -\gamma_0. \tag{1.31}$$

Proof. Choose a function $\chi \in C^\infty(\mathbb{R})$ supported in $t > 0$ and such that $\chi = 1$ for $t > \frac{1}{2}\underline{t} := t_0$. Consider

$$u(t, x) = \chi(t) e^{i(t\tau + x\xi)} r$$

with $(\tau, \xi) \in \mathbb{C} \times \mathbb{R}^d$ such that $\operatorname{Im} \tau < 0$ and $p(\tau, \xi) = 0$ and r satisfying $L(\tau, \xi) r = 0$ and $|r| = 1$. In particular, $Lu = 0$ when $t > t_0$ and (1.30) implies that, with a new constant C,

$$e^{-\underline{t} \operatorname{Im} \tau} \leq C(1 + |\tau|^2 + |\xi|^2)^{s/2}. \tag{1.32}$$

Let $\mu(\sigma) = \sup(-\operatorname{Im} \tau)$ where the supremum is taken for the $\tilde{\xi} = (\tau, \xi) \in \mathbb{C} \times \mathbb{R}^d$ such that $p(\tilde{\xi}) = 0$ and , $\operatorname{Im} \tau < 0$ and $|\tilde{\xi}| = \sigma$. The estimate above implies that $\mu(\sigma)$ grows at most logarithmically as $\sigma \to \infty$. By a lemma on sub-algebraic functions (see e.g. Corollary A.2.6 in [Hör]), this implies that μ is bounded and (1.31) follows. □

1.5 Properties of hyperbolic polynomials

A very important feature of hyperbolic equations is the finite speed of propagation. It is closely related to the property that the direction of time can be perturbed. This leads to give definitions independent of coordinates. So we change slightly the notations and we denote by $\tilde{x} \in \mathbb{R}^{1+d}$ the time-space variables and by $\tilde{\xi}$ the dual variables. We consider $N \times N$ first order system systems $\sum_{j=0}^d A_j \partial_{\tilde{x}_j} + B$. Their characteristic determinant is $p(\tilde{\xi}) = \det\big(\sum_{j=0}^d i\tilde{\xi}_j A_j + B\big)$, the principal part of which is $p_0(\tilde{\xi}) = \det\big(\sum_{j=0}^d i\tilde{\xi}_j A_j\big)$

Definition 1.13. *A polynomial* $p(\tilde{\xi})$ *with principal part* p_0 *is said to be hyperbolic in the direction* ν *if* $p_0(\nu) \neq 0$ *and there is* γ_0 *such that* $p(i\tau\nu + \tilde{\xi}) \neq 0$ *for all* $\xi \in \mathbb{R}^{1+d}$ *and all real* $\tau < -\gamma_0$.

A first order system $L = \sum_{j=0}^{d} A_j \partial_{\tilde{x}_j} + B$ *is said to be hyperbolic in the direction* $\nu \in \mathbb{R}^{1+d}$ *if the characteristic determinant is.*

Theorem 1.14. *i) If* p *is hyperbolic in the direction* ν*, then it is also hyperbolic in the direction* $-\nu$*. In particular, there is* γ_1 *such that the roots in* τ *of* $p(\tilde{\xi} + \tau\nu) = 0$ *are located in* $|\operatorname{Im}\tau| \leq \gamma_1$.

ii) If p *is hyperbolic in the direction* ν *then* p_0 *is also hyperbolic in this direction. This is equivalent to the conditions that for all* $\tilde{\xi}$*, the roots in* τ *of* $p(\tau\nu + \tilde{\xi}) = 0$ *are real.*

iii) If f p *is hyperbolic in the direction* ν *and if* Γ *denotes the component of* ν *in the open set* $\{p(\tilde{\xi}) \neq 0\}$*, then* Γ *is an open convex cone in* $\mathbb{R}^{1+d}$ *and* p *is hyperbolic in any direction* $\vartheta \in \Gamma$.

Proof. See Gårding [Gar] or Hörmander [Hör]. □

In coordinates (t, x) where $\nu = dt = (1, 0, \ldots, 0)$, we just recover the Definition 1.2.

There is also a similar definition of strong hyperbolicity:

Definition 1.15. $L = \sum_{j=0}^{d} A_j \partial_{\tilde{x}_j} + B$ *is strongly hyperbolic in the direction* ν *if and only if for all matrix* B_1*,* $L + B_1$ *is hyperbolic in the direction* ν.

This definition depends only on the principal part L_0 of L. Theorem 1.7 can be reformulated as follows

Theorem 1.16. $L = \sum_{j=0}^{d} A_j \partial_{\tilde{x}_j}$ *is strongly hyperbolic in the direction* ν *if and only if one of the following condition is satisfied*

i) there is a constant C *such that for all* $(\gamma, \tilde{\xi}, u) \in \mathbb{R} \times \mathbb{R}^{1+d} \times \mathbb{C}^N$*:*

$$\big|\gamma u\big| \leq C \big|L(\tilde{\xi} + i\gamma\nu)u\big|. \tag{1.33}$$

ii) There is a real C_1 *such that*

$$\forall t \in \mathbb{R}, \forall \tilde{\xi} \in \mathbb{R}^{1+d} \ : \qquad \big|e^{itA(\tilde{\xi})}\big| \leq C_1. \tag{1.34}$$

iii) All the the eigenvalues λ *of* $A(\tilde{\xi})$ *are real and semi-simple and there is a real* C_2 *such that all the eigen-projectors* $\Pi_\lambda(\tilde{\xi})$ *satisfy*

$$\forall \tilde{\xi} \ : \qquad \big|\Pi_\lambda(\tilde{\xi})\big| \leq C_2. \tag{1.35}$$

iv) There are definite positive matrices $S(\tilde\xi)$ and there are constants C_4 and $c_4 > 0$ such that for all $\tilde\xi$, $S(\tilde\xi)A(\tilde\xi)$ is symmetric, and

$$\big|S(\tilde\xi)\big| \le C_4, \qquad S(\tilde\xi) \ge c_4 \mathrm{Id}. \tag{1.36}$$

Another important property is that the strong form of hyperbolicity is preserved for all $\vartheta \in \Gamma$.

Theorem 1.17. *If L is strongly hyperbolic in the direction ν, then it is strongly hyperbolic in any direction $\vartheta \in \Gamma$. Moreover, for all compact cone $\Gamma_1 \subset \Gamma$ with compact bases, there is a constant C such that*

$$\mathrm{Im}\,\tilde\xi \in \Gamma_1 \qquad \Rightarrow \qquad |\mathrm{Im}\,\tilde\xi||u| \le C|L_0(\tilde\xi)u| \tag{1.37}$$

Proof. This is a consequence of the fact that the cone of hyperbolicity Γ depends only the principal part L_0. Thus if $L + B$ is hyperbolic in the direction ν for all B, then $L + B$ is hyperbolic in the direction ϑ for all B if $\vartheta \in \Gamma$. □

1.6 Finite speed of propagation

Theorem 1.18. *If L is hyperbolic in directions $\nu \in \Gamma$, then L has a unique fundamental solution E supported in the polar cone of Γ .*

$$\Gamma^\circ = \{\tilde x \in \mathbb{R}^{1+d}\ ,\ \forall \tilde\xi \in \Gamma\ :\ \tilde\xi \cdot \tilde x \ge 0\ \} \tag{1.38}$$

Proof. By Proposition 1.6, for all $\nu \in \Gamma$, there is a fundamental solution E_ν supported in $\{\tilde\xi \cdot \tilde x \ge 0\}$. By deformation, using the definition of $|Gamma$ Holmgren' uniqueness theorem implies that they all coincide and therefore E is supported in the intersection of the half spaces $\{\tilde\xi \cdot \tilde x \ge 0\}$.

One can also give a more constructive proof. Fix $\nu \in \Gamma$ and use coordinates such that $\nu = dt$. The fundamental solution constructed in Proposition 1.6 can be written

$$E(\tilde x) = \frac{1}{(2\pi)^{d+1}} \int_{\{\mathrm{Im}\,\tilde\xi \cdot \tilde x = \gamma\}} e^{i\tilde\xi\cdot\tilde x} L(\tilde\xi)^{-1} d\tilde\xi, \tag{1.39}$$

where the integral is understood as an inverse Fourier transform. The matrix $L(\tilde\xi)^{-1}$ is defined and holomorphic for $\tilde\xi \in \mathbb{R}^{1+d} - i\Gamma_1^R$, if Γ_1 is a subcone with compact base in Γ and $\Gamma_1^R = \{\tilde\eta \in \Gamma^1; |\eta| \ge R\}$, provided that R is large enough. By Paley-Wiener theorem, E is supported in $\{\tilde x \cdot \vartheta \ge 0\}$ for all $\vartheta \in \Gamma_1\}$. □

Let us come back to coordinates (t, x) such that $dt \in \Gamma$. As a corollary we obtain :

Theorem 1.19. *If $u_0 \in \mathcal{D}'(\mathbb{R}^d)$ [resp $C^\infty(\mathbb{R}^d)$], the Cauchy problem $Lu = 0, u_{|t=0} = u_0$ has a unique solution u continuous in times with values in $\mathscr{D}'$ [resp. $C^\infty(\mathbb{R}^{1+d})$] and*

$$\operatorname{supp}(u) \subset \operatorname{supp}(u_0) + \Gamma^\circ. \tag{1.40}$$

Proof. The construction above shows that E is continuous with values in $\mathscr{E}'$ the space of distributions with compact support so the following definition makes sense:

$$u(t, \cdot\,) = E(t, \cdot\,) * u_0(\,\cdot\,) \tag{1.41}$$

The theorem follows. □

There are similar results for equations with source terms $f \neq 0$.

2 Boundary value problems

2.1 Introduction

Consider

$$\begin{cases} Lu = f, & x_n > 0, \\ Mu_{|x_n=0} = g. \end{cases} \tag{2.1}$$

Here $x_n = n \cdot \tilde{x}$ and $A_n = L(n)$ is invertible. L is assumed to be hyperbolic. The matrices A_j and $L(\tilde{\xi})$ act from spaces $\mathbb{E}$ to $\mathbb{F}$ and M from $\mathbb{E}$ to $\mathbb{G}$. We assume in this lecture that the boundary is not characteristic, that is that

$$\det L_0(n) \neq 0, \tag{2.2}$$

where L_0 is the principal part of L.

At the end, we want to solve the problem (2.1) for positive time (a direction of hyperbolicity) with an initial datum at $t = 0$ (the initial boundary value problem, in short IBVP). An intermediate step is to solve the equation for t running from $-\infty$ to $+\infty$ (that is in $\mathbb{R}^{1+d}_+ = \{x_n \geq 0\}$), in spaces of functions or distributions which are allowed to have an exponential growth in time at $+\infty$, but still decaying (temperate) at infinity in space. More precisely, given a direction of hyperbolicity ν, supposed to be independent of n, we set $t = \nu \cdot \tilde{x}$ and we fix coordinates $\tilde{x} = (t, x', x_n)$. We look for solutions of the form

$$e^{\gamma t} u_\flat(t, x) \tag{2.3}$$

with $u_\flat$ tempered. The equations for $u_\flat$ reads

$$\begin{cases} L_\gamma u_\flat = f_\flat & x_n > 0, \\ M u_{\flat | x_n = 0} = g_\flat, \end{cases} \tag{2.4}$$

where

$$L_\gamma(\partial) = L(\partial) + \gamma L(\nu), \qquad L_\gamma(\tau, \xi) = L(\tau - i\gamma, \xi). \tag{2.5}$$

So the first goal is to solve (2.4) when γ is large enough, say $\gamma \geq \gamma_0$, and next to draw conclusions for (2.1) and for the IBVP.

Objectives:

- Introduce the stability condition for (2.4), the Lopatinski condition;
- Introduce the method of symmetrizers ;
- Discuss the causality principle;
- Discuss the finite speed propagation property in relation to the choice of the time direction.

2.2 The basic bvp for o.d.e

Apply the tangential Fourier Laplace transform to (2.1), that is the Fourier transform with respect to (t, x') to (2.4). To simplify notations, we call u the resulting function. The equations are

$$\begin{cases} \partial_{x_n} u + iG(\zeta) u = f, & x_n > 0, \\ M u_{| x_n = 0} = g. \end{cases} \tag{2.6}$$

Here $\zeta = (\tau, \xi') \in \mathbb{C} \times \mathbb{R}^{d-1}$, $\operatorname{Im} \tau = -\gamma < 0$ and

$$G(\zeta) = L_0(n)^{-1} L(\zeta, 0). \tag{2.7}$$

Lemma 2.1. *Hyperbolicity implies that there is γ_0 such that for $\operatorname{Im} \tau < -\gamma_0$, $G(\zeta)$ has no real eigenvalue.*

Proof. If G has a real eigenvalue λ, then $\xi_n = -\lambda$ satisfies $\det L(\tau, \xi', \xi_n) = 0$, which requires that $|\mathrm{Im}\,\tau| \leq \gamma_0$ for some γ_0. □

Definition 2.2. *For* $\mathrm{Im}\,\tau < 0$*, the incoming space* $\mathbb{E}^{in}(\zeta)$ *[resp. outgoing space* $\mathbb{E}^{out}(\zeta)$ *] is the invariant space of* $G(\zeta)$ *associated to the eigenvalues in* $\{\mathrm{Im}\,\lambda < 0\}$ *[resp.* $\{\mathrm{Im}\,\lambda > 0\}$*]. We denote by* Π^{in} *[resp.* Π^{out}*] the spectral projectors on these spaces.*

Lemma 2.3. *The dimension of* $\mathbb{E}^{in}$ *is equal to* N_+*, the number of positive eigenvalues of* $L_0(n)^{-1}L_0(\nu)$*.*

Proof. This number is independent of ζ. We compute it for $\zeta = (-i\gamma, 0)$ with $\gamma \to +\infty$. Indeed,

$$G(-i\gamma, 0) = -i\gamma G_\gamma, \qquad G_\gamma := L_0(n)^{-1}L_0(\nu) + \gamma^{-1}B.$$

Thus By hyperbolicity and (2.2), the eigenvalues of $L_0(n)^{-1}L_0(\nu)$ are real and $\neq 0$. Thus, for γ large, the eigenvalues of G_γ split into two groups. N_+ of them are in $\mathrm{Re}\,\lambda > 0$ and $N - N_+$ are in $\mathrm{Re}\,\lambda < 0$. Hence G_γ has N_+ eigenvalues in $\mathrm{Im}\,\mu < 0$ and $N - N_+$ in $\mathrm{Im}\,\mu < 0$. □

We now consider the o.d.e. $(\partial_{x_n} + iG)u = f$ in spaces of temperate (or decaying) functions on $[0, +\infty[$. By Lemma 2.1 the solutions of the homogeneous equations $u = e^{-ix_nG}a$, split into groups, those which decay exponentially at $+\infty$ when $a \in \mathbb{E}^{in}$ and those which decay exponentially at $-\infty$ when $a \in \mathbb{E}^{out}$. One has the following representation:

$$e^{-ix_nG}\Pi^{in} = \frac{1}{2i\pi}\int_{\mathcal{C}^+} e^{ix_n\xi_n}\big(\xi_n\mathrm{Id} + G\big)^{-1}d\xi_n \tag{2.8}$$

where $\mathcal{C}^+$ is a contour in $\{\mathrm{Im}\,\xi_n > 0\}$ surrounding the spectrum of $-G$ located in this half space. Similarly

$$e^{-ix_nG}\Pi^{out} = \frac{1}{2i\pi}\int_{\mathcal{C}^-} e^{ix_n\xi_n}\big(\xi_n\mathrm{Id} + G\big)^{-1}d\xi_n, \tag{2.9}$$

with $\mathcal{C}^- \subset \{\mathrm{Im}\,\xi_n < 0\}$.

Lemma 2.4. $e^{-ix_nG}\Pi^{in}$ *[resp.* $e^{-ix_nG}\Pi^{out}$ *is exponentially decaying when* $x_n \to +\infty$ *[resp.* $x_n \to -\infty$*]. If* f *is temperate at* $+\infty$*, the temperate solutions of* $(\partial_{x_n} + iG)u = f$ *on* $\mathbb{R}_+$ *are*

$$u(x_n) = e^{-ix_nG}a + If(x_n), \qquad a \in \mathbb{E}^{in} \tag{2.10}$$

where

$$I f(x_n) = \int_0^{x_n} e^{i(y_n - x_n)G} \Pi^{in} f dy_n \\ - \int_{x_n}^{\infty} e^{i(y_n - x_n)G} \Pi^{out} f dy_n. \tag{2.11}$$

Therefore, to solve (2.6) is remains to check the boundary condition, that is to solve for $a = \Pi^{in} u_0$

$$a \in \mathbb{E}^{in}(\zeta), \qquad Ma = g - MI(f)_{|x_n=0} \tag{2.12}$$

Proposition 2.5. *For* $\operatorname{Im}\tau < -\gamma_0$*, the boundary value problem* (2.6) *has a unique (temperate) [resp. in the Schwartz class] [resp. in* L^2*] solution for all* f *in the same space and all* $g \in \mathbb{G}$*, if and only if* $M_{|\mathbb{E}^{in}}$ *is an isomorphism from* $\mathbb{E}^{in}$ *to* $\mathbb{G}$*.*

This leads to the natural condition which we assume to be satisfied from now on.

Assumption 2.6. *The number of boundary conditions is* N_+*, i.e. the boundary operator acts from* $\mathbb{E}$ *to* $\mathbb{G}$ *where* $\dim \mathbb{G} = N_+$*.*

The analysis above also legitimates the following condition:

Definition 2.7. *We say that the* (2.1) *satisfies Lopatinski condition (in the time direction dt) if there is* γ_0 *such that for all* $\zeta = (\tau, \xi')$ *with* $\operatorname{Im}\tau < -\gamma_0$*,* $\mathbb{E}^{in}(\zeta) \cap \ker M = \{0\}$*.*

2.3 Fourier synthesis

To get solutions for (2.4), we must be able to perform the inverse Fourier transform, that is we need estimates. For simplicity, we give details in L^2 spaces.

We use the representation (2.10) of the solution

$$\hat{u}(x_n, \tau, \xi') = e^{-ix_n G(\zeta)} \hat{a}(\zeta) + I(\zeta, \hat{f}(\,\cdot\,, \tau, \xi')) \tag{2.13}$$

where $I(\zeta, \hat{f})$ is given by (2.15) and $\zeta = (\tau, \xi')$ with $\operatorname{Im}\tau < -\gamma_0$ for some γ_0.

Lemma 2.8. *There are* $m_0 \geq 1$*,* $\gamma_0 \geq 0$ *and* C *such that for all real* ξ_n *and all* ζ *with* $\operatorname{Im}\tau < -\gamma_0$

$$\gamma^{m_0} \big| (\xi_n \mathrm{Id} + G(\zeta)^{-1} \big| \leq \langle \zeta \rangle^{m_0 - 1}. \tag{2.14}$$

Proof. When $|\xi_n| \le C\langle\zeta\rangle$ this is the resolvent estimate, and when $|\xi_n|$ is large, there is a bound in $O(|\xi_n|^{-1})$. □

Lemma 2.9. *$f \in L^2(\mathbb{R}_+)$ then $I(f)$ is the restriction to $\mathbb{R}_+$ of the solution in L^2 of $(\partial_{x_n} + iG)\tilde u = \tilde f$ where $\tilde f$ is the extension of f by 0 on the negative axis.*

Proof. $\tilde u$ is given by the formula

$$\tilde u(x_n) = \int_{-\infty}^{x_n} e^{i(y_n - x_n)G}\Pi^{in}\tilde f dy_n - \int_{x_n}^{\infty} e^{i(y_n - x_n)G}\Pi^{out}\tilde f dy_n. \tag{2.15}$$

□

Corollary 2.10. *There are C and γ_0 such that when $\operatorname{Im}\tau < -\gamma_0$*

$$\gamma^{m_0}\big\|I(f)\big\|_{L^2} \le C\langle\zeta\rangle^{m_0-1}\big\|f\big\|_{L^2}, \tag{2.16}$$

$$\gamma^{m_0}\big|I(f)_{|x_n=0}\big| \le C\langle\zeta\rangle^{m_0-\frac12}\big\|f\big\|_{L^2}. \tag{2.17}$$

Proof. $\tilde u$ can be computed using a Fourier transform in x_n: its Fourier transform is

$$\hat u(\xi_n) = -i(\xi_n + G)^{-1}\hat f$$

where $\hat f$ is the Fourier transform of $\tilde f$. The L^2 estimate of $\tilde u$ follows from (2.14). The second estimate follows using the equation and the inequality

$$|\tilde u(0)|^2 \le 2\big\|\tilde u\big\|_{L^2}\big\|\partial_{x_n}\tilde u\big\|_{L^2} \le 2\big\|\tilde u\big\|_{L^2}\big\|\tilde f\big\|_{L^2} + O(\langle\zeta\rangle)\big\|\tilde u\big\|_{L^2}^2. \tag{2.18}$$

□

For the first term in (2.13), we use the following estimate.

Lemma 2.11. *There is C such that for $\operatorname{Im}\tau < -\gamma_0$ and $a \in \mathbb{E}^{in}(\zeta)$, $u = e^{-ix_nG}a$ satisfies*

$$\gamma^{m_0}\big\|u\big\|_{L^2(\mathbb{R}_+)} \le \langle\zeta\rangle^{m_0-1}|a|. \tag{2.19}$$

Proof. Introduce $L^* = -\partial_x - iG^*$ the adjoint of $L = \partial_x + iG$. Then

$$\big(Lu, v\big)_{L^2(\mathbb{R}_+)} - \big(u, L^*v\big)_{L^2(\mathbb{R}_+)} = -\big(u(0), v(0)\big). \tag{2.20}$$

In particular, if $u = e^{-ix_nG}a$ with $a \in \mathbb{E}^{in}$, one has

$$\big(u, L^*v\big)_{L^2(\mathbb{R}_+)} = \big(a, v(0)\big). \tag{2.21}$$

For $f \in L^2(\mathbb{R}_+)$, extend it by 0 for negative x_n and consider the solution v of $L^*v = \tilde{f}$. L^* satisfies the same estimate (2.14) as L and repeating the proof of the Corollary above, we obtain the estimate

$$\gamma^{m_0}|v(0)| \leq C\langle\zeta\rangle^{m_0-\frac{1}{2}}\big\|f\big\|_{L^2}. \tag{2.22}$$

With (2.20), this implies (2.19). □

Next we need estimates for the solutions of the equation (2.12). The Lopatinski condition says that there is an inverse mapping $R(\zeta) : \mathbb{G} \mapsto \mathbb{E}^{in}(\zeta)$ such that $MR(\zeta) = \mathrm{Id}_{\mathbb{G}}$.

Lemma 2.12. *If the Lopatinski condition is satisfied, there are* γ_1*,* m *and* C *such that for* $\mathrm{Im}\,\tau \leq -\gamma_1$

$$a \in \mathbb{E}^{in}(\zeta) \quad \Rightarrow \quad |\mathrm{Im}\,\tau|^m\big|u\big| \leq C\langle\zeta\rangle^m \ \big|Ma\big|. \tag{2.23}$$

Equivalently, this means that

$$\big|R(\zeta)\big| \leq C|\mathrm{Im}\,\tau|^m/\langle\zeta\rangle^m. \tag{2.24}$$

Proof. Again, the polynomial bound depends on properties of semi-algebraic functions. See Appendix 2. □

Summing up, we have proved the following:

Theorem 2.13. *Suppose that the system is hyperbolic in the time direction and the Lopatinski condition is satisfied. Then, there are* C*,* m *and* γ_0 *such that, when* $\mathrm{Im}\,\tau < -\gamma_0$*, for all* $f \in L^2(\mathbb{R}_+)$ *and all* $g \in \mathbb{C}^{N_+}$*, the problem* (2.6) *has a unique solution* $u \in H^1(\mathbb{R}_+)$ *wich satisfies,*

$$\gamma\big\|u\big\|^2_{L^2} + \big|u(0)\big|^2 \leq C\big(\langle\zeta\rangle/\gamma\big)^m\big(\gamma^{-1}\big\|f\big\|^2_{L^2} + \big|g\big|^2\big). \tag{2.25}$$

where $\gamma = -\mathrm{Im}\,\tau$*.*

By Fourier inversion, we obtain the following corollary.

Theorem 2.14. *If the Lopatinski condition is satisfied, then there are* γ_0 *such that for* $\gamma \geq \gamma_0$*,* $\sigma \geq 0$ $f \in H^{\sigma+m}(\mathbb{R}^{1+d}_+)$*,* $g \in H^{\sigma+m}(\mathbb{R}^d)$*, then the problem* (2.4) *has a unique solution* $u \in H^\sigma(\mathbb{R}^{1+d}_+)$*.*

Equivalently, for $f \in e^{\gamma t}H^{\sigma+m}(\mathbb{R}^{1+d}_+)$*,* $g \in e^{\gamma t}H^{\sigma+m}(\mathbb{R}^d)$*, the problem* (2.1) *has a unique solution* $u \in e^{\gamma t}H^\sigma(\mathbb{R}^{1+d}_+)$*.*

2.4 The method of symmetrizers

Estimates for symmetric hyperbolic BVP are easily obtained by integrations by part. The method of symmetrizers is also a key ingredient in the analysis of systems with variable coefficients. What we introduce can be seen as the symbolic part of the analysis, see e.g. [Kr].

Definition 2.15. *A symmetrizer is $S(\zeta)$ such that*

$$S(\zeta) = S(\zeta)^*, \qquad \operatorname{Im} S(\zeta)G(\zeta) \geq c(\zeta)\mathrm{Id}, \qquad |S(\zeta)| \leq C \tag{2.26}$$

with $c(\zeta) > 0$. The boundary condition M is dissipative [resp. strictly dissipative] for S if

$$S(\zeta) \geq 0 \qquad [\textit{resp. } S(\zeta) \geq c_1(\zeta)\mathrm{Id}\ ] \quad \textit{on } \ker M. \tag{2.27}$$

Proposition 2.16. *If S is a symmetrizer and M is strictly dissipative, then the Lopatinsky condition is satisfied. The equation* (2.6) *is well posed in L^2 and the solutions satisfy*

$$c\big\|u\big\|^2_{L^2} + c_1\big|u(0)\big|^2 \lesssim \frac{1}{c}\big|f\big\|^2_{L^2} + \frac{1}{c_1}\big|g\big|^2 \tag{2.28}$$

Proof. For decaying solutions, one has the energy balance

$$2\mathrm{Re}\,\big(Sf, u\big)_{L^2} = -(Su(0), u(0)) - 2\mathrm{Im}\,\big(SGu, u\big)_{L^2} \tag{2.29}$$

and

$$c\big\|u\big\|^2_{L^2} + \frac{1}{2}(Su(0), u(0)) \lesssim \frac{1}{c}\big|f\big\|^2_{L^2}. \tag{2.30}$$

In particular, if $f = 0$, this implies that

$$S \leq 0 \qquad on \quad \mathbb{E}^{in}. \tag{2.31}$$

Hence, strict dissipativity implies that $\mathbb{E}^{in} \cap \ker M = \{0\}$ and the Lopatinski condition is satisfied.

Let $\mathbb{H}_1$ be a fixed space such that $\mathbb{E} = \ker M \oplus \mathbb{H}_1$. Let C_1 be such that

$$u \in \mathbb{H}_1 \qquad \Rightarrow \qquad |u| \leq C_1|Mu|. \tag{2.32}$$

Decompose $u \in \mathbb{E}$ into $u = u_0 + u_1 \in \ker M \oplus \mathbb{H}_1$. Then

$$\begin{aligned} (Su, u) &= (Su_0, u_0) - O(|u_1|^2) - O(|u_1||u_0|) \\ &\geq \frac{1}{2}c_1|u_0|^2 - \frac{C}{c_1}|u_1|^2 \geq \frac{1}{4}|u|^2 - \frac{C'}{c_1}|Mu|^2 \end{aligned} \tag{2.33}$$

since $Mu_1 = Mu$. This proves (2.28) and the proposition follows. □

If the boundary condition is only dissipative, then the conclusion is that

$$Mu(0) = 0 \qquad \Rightarrow \qquad c\|u\|_{L^2}^2 \lesssim \frac{1}{c}\|f\|_{L^2}^2. \tag{2.34}$$

Given an inhomogeneous boundary term g, we can choose $a \in \mathbb{E}$ such that $Ma = g$ and $|a| \lesssim |g|$. We apply the estimate above to $u - a^{-\delta x_n}$ and obtain that

$$c\|u\|_{L^2}^2 \lesssim \frac{1}{c}\|f\|_{L^2}^2 + (\frac{c}{\delta} + \frac{\delta^2 + \langle\zeta\rangle^2}{c\delta})|g|^2 \tag{2.35}$$

Choosing $\delta \approx \langle\zeta\rangle$, we get that

$$c\|u\|_{L^2}^2 \lesssim \frac{1}{c}\|f\|_{L^2}^2 + (\frac{c}{\langle\zeta\rangle} + \frac{\langle\zeta\rangle}{c})|g|^2. \tag{2.36}$$

An estimate of $u(0)$ and be deduced from the inequality

$$\big|u(0)\big|^2 \leq 2\|\partial_{x_n}u\|_{L^2}\|u\|_{L^2} \lesssim \|f\|_{L^2}\|u\|_{L^2} + \langle\zeta\rangle\|u\|_{L^2}^2. \tag{2.37}$$

In particular, if $f = 0$ and $g = 0$, then $u = 0$. Hence we have proved

Proposition 2.17. *If S is a symmetrizer and M is dissipative, then the Lopatinsky condition is satisfied. The equation* (2.6) *is well posed in L^2 and the solutions satisfy* (2.36) *and* (2.37)*.*

2.5 Dissipative symmetric hyperbolic BVP

Important examples are dissipative BVP for symmetric hyperbolic systems in the sense of Friedrichs (see the definition at Example 1.10). If S is a Friedrichs' symmetrizer, then $-S$ is a symmetrizer for the o.d.e in the sense of definition 2.15. Accordingly,

Definition 2.18. *If L is symmetric hyperbolic in the sense of Friedrichs with symmetrizer S, the boundary condition M is said to be dissipative [resp. strictly dissipative] when $SL_0(n) \leq 0$ [resp. $SL_0(n) \ll 0$] on $\ker M$.*

It is maximal, dissipative or strictly dissipative, if in addition $\dim\ker M = N - N_+$.

If the condition is dissipative, then $\dim\ker M \leq N - N_+$ since the signature of $SL_0(n)$ is $(N_+, N - N_+)$. This explains the terminology "maximal dissipative".

We recall briefly the mains results of this theory. They are based on the following the energy balance, where we use coordinates (t,x):

$$\begin{aligned}\int_{\mathbb{R}^d_+}\big(SA_0u(T),u(T)\big)dx-\int_{[0,T]\times\mathbb{R}^{d-1}}\big(SA_nu_0,u_0\big)dtdx'\\ =\int_{\mathbb{R}^d_+}\big(SA_0u(0),u(0)\big)dx+2\mathrm{Re}\int_{[0,T]\times\mathbb{R}^d_+}\big(SLu,u\big)dtdx.\end{aligned}\tag{2.38}$$

Proposition 2.19. *Consider a symmetric hyperbolic system.*

i) If the boundary conditions are dissipative and if u satisfies the homogeneous boundary conditions $Mu=0$ on the boundary,

$$\big\|u(t)\big\|_{L^2}\leq Ce^{\gamma t}\big\|u(0)\big\|_{L^2}+C\int_0^t e^{\gamma(t-s)}\big\|Lu(s)\big\|_{L^2}ds.\tag{2.39}$$

If the boundary condition is maximal dissipative, then for all $f\in L^1([0,T],L^2)$ and $u_0\in L^2$, the initial boundary value problem $Lu=f$, $u_{|t=0}=u_0$, $Mu_{|x_n=0}=0$ has a unique solution $u\in C^0([0,T];L^2)$ which satisfies (2.39)*.*

ii) If the boundary conditions are strictly dissipative then $u_\gamma=e^{-\gamma t}u$ satisfies

$$\begin{aligned}\big\|u_\gamma(t)\big\|_{L^2}+\big\|u_{\gamma|x_n=0}\big\|_{L^2([0,t]\times\mathbb{R}^{d-1}}\leq C\big\|u(0)\big\|_{L^2}\\ +C\int_0^t\big\|e^{-\gamma s}Lu(s)\big\|_{L^2}ds+C\big\|u_{\gamma|x_n=0}\big\|_{L^2([0,t]\times\mathbb{R}^{d-1}}\end{aligned}\tag{2.40}$$

If the boundary condition is maximal dissipative, then for all $f\in L^1([0,T],L^2)$, $g\in L^2[0,T]\times\mathbb{R}^{d-1}$ and $u_0\in L^2$, the initial boundary value problem $Lu=f$, $u_{|t=0}=u_0$, $Mu_{|x_n=0}=0$ has a unique solution $u\in C^0([0,T];L^2)$ which satisfies (2.40)*.*

In the maximal dissipative cases one can also solve the inhomogeneous boundary value problem, but, in general, not for general $g\in L^2$ if one want a L^2 solution ($g\in H^{\frac12}$ is sufficient), and one does not recover the L^2 estimate of the trace $u_{|x_n=0}$, only and $H^{-\frac12}$ estimate which is only a consequence of the fact that $u\in L^2$, $Lu\in L^2$ and the boundary is not characteristic.

Note that the "semi group" estimates above (meaning in $C^0([0,T];L^2)$) imply estimates in $L^2([0,T];L^2)$. For instance, (2.40) implies that for $\gamma\geq\gamma_0$ and $u\in e^{\gamma t}\mathscr{S}(\overline{\mathbb{R}}^{1+d}_+)$:

$$\gamma\big\|u\big\|^2_{L^2_\gamma}+\big\|u_{|x_n=0}\big\|^2_{L^2_\gamma}\lesssim\frac{1}{\gamma}\big\|Lu\big\|^2_{L^2_\gamma}+\big\|Mu_{|x_n=0}\big\|^2_{L^2_\gamma}\tag{2.41}$$

where $L^2_\gamma = e^{\gamma t} L^2$. Equivalently, $v = e^{-\gamma t} u$ satisfies

$$\gamma \big\| v \big\|^2_{L^2} + \big\| u_{|x_n=0} \big\|^2_{L^2} \lesssim \frac{1}{\gamma} \big\| (L + \gamma A_0) u \big\|^2_{L^2} + \big\| M u_{|x_n=0} \big\|^2_{L^2} \tag{2.42}$$

2.6 Maximal estimates and the uniform Lopatinski condition

In the constant coefficient case, by tangential Fourier transform, the estimate (2.42) is equivalent to the a similar estimate for the o.d.e (2.6) : for $\gamma \geq \gamma_0$, $\zeta = (\tau - i\gamma, \xi')$ and $u \in \mathscr{S}(\overline{\mathbb{R}}_+)$

$$\gamma \big\| u \big\|^2_{\mathbb{R}_+} + \big| u(0) \big|^2 \leq C \big(\gamma^{-1} \big\| f \big\|^2_{L^2(\mathbb{R}_+)} + \big| g \big|^2 \big) \tag{2.43}$$

with $f = \partial_{x_n} u + i G(\zeta) u$ and $g = M u(0)$. The important point is that C is independent of ζ when $\operatorname{Im} \tau \leq -\gamma_0$.

Applied to solutions of $\partial_{x_n} u + i G(\zeta) u = 0$, this implies that

$$\forall u \in \mathbb{E}^{in}(\zeta), \qquad \big| u \big| \leq C \big| M u \big|. \tag{2.44}$$

Definition 2.20. *The uniform Lopatinski condition is said to be satisfied when the condition* $\dim \mathbb{G} = N_+$ *and there are constants* G *and* γ_0 *such that the estimate* (2.44) *is satisfied.*

The improvement with respect to the weak form of the condition is that the constant in C (2.44) can be taken independent of ζ.

Remark 2.21. The discussion before the definition shows that the uniform Lopatinski condition is necessary for the validity of the maximal estimates.

Proposition 2.22. *The uniform Lopatinski condition is satisfied for* M *if and only if there is* $\varepsilon > 0$ *such that the Lopatinski condition is satisfied for all* M' *such that* $|M - M'| \leq \varepsilon$.

Proof. If (2.44) is satisfied then it holds for M' with C replaced by $2C$ if $C|M - M'| \leq \frac{1}{2}$. Conversely, the condition implies that

$$u \in \mathbb{E}^{in}, \ |Mu| \leq \varepsilon |u| \quad \Rightarrow \quad u = 0$$

and hence (2.44) holds with $C = \varepsilon^{-1}$. □

Theorem 2.23. *If the maximal estimates are satisfied for some boundary conditions* M_0*, in particular if the system is symmetric in the sense of Friedrichs, then the uniform Lopatinski condition is necessary and sufficient for the validity of the maximal estimates.*

2.7 Kreiss symmetrizers

A major contribution to the theory has been given by O.Kreiss [Kr] who constructed tangential symmetrizers to prove that, for a class o the uniform Lopatinski condition is sufficient for the validity of the maximal estimates. Because zero-th order term are irrelevant, we assume here that $L = L_0$ is homogeneous. He proved the following.

Theorem 2.24. *Il the system L is striclty hyperbolic and the boundary conditions satisfy the uniform Lopatinski condition, then there are constants C and $c > 0$ and symmetrizers $S(\zeta)$ for $\zeta = (\tau - i\gamma, \xi')$, $\gamma > 0$, such that*

$$S(\zeta) = S(\zeta)^*, \qquad |S(\zeta)| \leq C \tag{2.45}$$

$$\operatorname{Im} S(\zeta)G(\zeta) \geq c\gamma \mathrm{Id} \tag{2.46}$$

$$S(\zeta) \geq c_1(\zeta)\mathrm{Id} \;] \quad \textit{on } \ker M. \tag{2.47}$$

Strictly hyperbolic means that the eigenvalues of $A(\xi)$ are real and simple. Note that in any case, strong hyperbolicity is necessary to have maximal estimates, as is it already necessary in the interior (see Theorem 1.7). The result is still true when the multiplicities of the eigenvalues are constant, and in some cases of variable multiplicities. See [Maj, Me3, MZ].

2.8 The causality principle

A weak form of the causality principle is that is u is a solution of the BVP (2.1) with data f and g which vanish in $t < t_0$, then $u = 0$ for $t < t_0$. This means that the values of a solution u at time t_0 only depend on the data for times $t \leq t_0$.

There is no loss of generality in assuming that $t_0 = 0$. For the solutions constructed by Fourier synthesis, the statement is clear because if the data vanish in the past, the Laplace Fourier transform has an holomorphic extension to a half space $\operatorname{Im} \tau < -\gamma_0$. This property is inherited by the solution, and together with the estimates we can conclude that $u = 0$ (see Appendix 2). For instance, we can state

Theorem 2.25. *With notations as in Theorem 2.14, if γ is larger than soem γ_0 , if $f \in e^{\gamma t} H^{\sigma+m}(\mathbb{R}^{1+d}_+)$, and $g \in e^{\gamma t} H^{\sigma+m}(\mathbb{R}^d)$ vanish for $t < 0$, then the problem (2.1) has a unique solution $u \in \bigcup_{\rho \geq \gamma} e^{\rho t} H^{\sigma}(\mathbb{R}^{1+d}_+)$. Moreover, u vanishes for $t < 0$ and belongs to $e^{\gamma t} H^{\sigma}(\mathbb{R}^{1+d}_+)$.*

2.9 Invariant definitions. The incoming spaces

Recall the notations.The symbol $L(\tilde{\xi}) = \sum \xi_j A_j - iB$ acts from $\mathbb{E}$ to $\mathbb{F}$ with $\dim \mathbb{E} = \dim \mathbb{F} = N$. We denote by $p(\tilde{\xi}) = \det L(\tilde{\xi})$. The principal symbol is $L_0(\tilde{\xi}) = \sum \tilde{\xi}_j A_j$. L is assumed to be hyperbolic in some direction ν and $\Gamma \subset \mathbb{R}^{1+d}$ denotes the open convex cone of hyperbolic directions. We consider the domain $\Omega = \{x_n > 0\}$ where $x_n = n \cdot x$, and $n \in \mathbb{R}^{1+d}$ is the inner conormal to the boundary. The boundary matrix is $A_n = L_0(n)$, supposed to be invertible, and we denotes by $G(\tilde{\xi}) = A_n^{-1} L(\tilde{\xi})$.

There is $\gamma_0 > 0$ such that

$$(2.48) \qquad \tilde{\xi} \in \mathbb{R}^{1+d}, \ \vartheta \in \Gamma \qquad \Rightarrow \qquad p(\tilde{\xi} - i\gamma_0 \nu - i\vartheta) \neq 0$$

(see [Gar] or Theorem 12.4.4 in [Hör]). We can normalize ν so that $\gamma_0 = 1$ so that, denoting by $\Gamma_\nu = \nu + \Gamma \subset \Gamma$,

$$(2.49) \qquad \operatorname{Im} \tilde{\xi} \in \Gamma_\nu \qquad \Rightarrow \qquad p(\tilde{\xi}) \neq 0.$$

This implies that for $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma_\nu$, $G(\tilde{\xi})$ has no real eigenvalue and hence the definition of incoming spaces has the following extension:

Definition 2.26. *For $\tilde{\xi} \in \mathbb{R}^{1+d} - i\Gamma_\nu$, the incoming space $\mathbb{E}^{in}(\tilde{\xi})$ is the invariant space of $G(\tilde{\xi})$ associated to the eigenvalues in $\{\operatorname{Im} \lambda < 0\}$.*

The dimension of $\mathbb{E}^{in}$ is constant, and was computed above.

Lemma 2.27. *$\mathbb{E}^{in}(\tilde{\xi})$ is an holomorphic vector bundle over $\mathbb{R}^{1+d} - i\Gamma_\nu$ of dimension N_+, the number of positive eigenvalues of $A_n^{-1} L(\nu)$.*

In particular, if $n \in \Gamma$ [resp. $n \in -\Gamma$], then $\mathbb{E}^{in} = \mathbb{C}^N$ [resp. $\mathbb{E}^{in} = \{0\}$]

From now on, we assume that $\pm n \notin \Gamma$ otherwise $\mathbb{E}^{in} = \mathbb{C}^N$ or $\mathbb{E}^{in} = \{0\}$ and all what follows is trivial.

Because

$$G(\tilde{\xi} + sn) = G(\tilde{\xi}) + s\mathrm{Id}$$

the incoming spaces have the property that

$$(2.50) \qquad \mathbb{E}^{in}(\tilde{\xi} + sn) = \mathbb{E}^{in}(\tilde{\xi})$$

if the segment $[\xi, \xi + sn]$ is contained in $\mathbb{R}^{1+d} - i\Gamma_\nu$. (This is trivial if $s \in \mathbb{R}$; if s is complex, the assumption is that for $t \in [0,1]$ the eigenvalues of $G(\tilde{\xi} + tsn)$ do not cross the real axis, implying that the invariant space associated to the eigenvalues in $\{\operatorname{Im} \lambda < 0\}$ is constant).

Consider the projection $\varpi : \mathbb{R}^{1+d} \mapsto \mathbb{R}^{1+d}/\mathbb{R}n \approx T^*\partial\Omega$ and its complex extension $\mathbb{C}^{1+d} \mapsto \mathbb{C}^{1+d}/\mathbb{C}n \approx \mathbb{C} \otimes T^*\partial\Omega$. Let $\Gamma^\flat$ denote the projection of Γ:

$$\Gamma^\flat = \{\zeta \ : \ \exists\tilde\xi \in \Gamma \ , \ \zeta = \varpi\tilde\xi\} \subset T^*\partial\Omega\backslash\{0\}. \tag{2.51}$$

It is an open convex cone in $T^*\partial\Omega$. Let $\Gamma^\flat_\nu = \nu^\flat + \Gamma^\flat = \varpi\Gamma_\nu$. It is convex and for $\zeta \in \Gamma^\flat_\nu$, $\varpi^{-1}(\zeta)$ is a segment in Γ. Thus the invariance (2.50) implies that $\mathbb{E}^{in}$ depends only on $\varpi\tilde\xi$ and legitimates the following definition:

Definition 2.28. *For $\zeta \in T^*\partial\Omega - i\Gamma^\flat_\nu$, we set*

$$\mathbb{E}^{in}(\zeta) = \mathbb{E}^{in}(\tilde\xi), \qquad \xi \in \mathbb{R}^{1+d} - i\Gamma_\nu, \quad \varpi\tilde\xi = \zeta. \tag{2.52}$$

In coordinates (t, x', x_n) with dual variables $(\tau, \xi', \xi_n) \in \mathbb{R}^d \times \mathbb{R}$, one can identify $T^*\partial\Omega$ with the first factor $\mathbb{R}^d$. This is what we did in the previous sections, and this is why we use the notation ζ for element of $T^*\partial\Omega$. More importantly, we have extended the definition of $\mathbb{E}^{in}$ to the complex domain $\{\operatorname{Im}\zeta \in \Gamma^\flat_\nu\}$.

When $L = L_0$ is homogeneous, then $\mathbb{E}^{in}$ is clearly homogeneous of degree 0 and defined in $\mathbb{R}^{1+d} - i\Gamma$. In general, because L_0 is hyperbolic with the same cone of hyperbolic directions Γ, we can introduce the incoming spaces associated to L_0, which we denote by. $\mathbb{E}^{in}_0(\tilde\xi)$. For $\tilde\xi \in \mathbb{R}^{1+d} - i\Gamma$ and $\varepsilon > 0$ small, we have

$$\begin{aligned}\Pi^{in}(\tilde\xi/\varepsilon) =& \frac{1}{2i\pi}\int_{\mathcal{C}^+} (z + G_0(\tilde\xi) - i\varepsilon A_n^{-1}B)^{-1} dz \\ &\to \Pi^{in}_0(\tilde\xi) \qquad \text{as } \varepsilon \to 0.\end{aligned} \tag{2.53}$$

This property is still true in the quotient $\tilde\xi \mapsto \zeta$. Note that these convergences hold for $\operatorname{Im}\zeta \in \Gamma^\flat$, which means in particular that $\operatorname{Im}\tilde\zeta \neq 0$. No uniformity in $\operatorname{Im}\zeta$ is claimed as $\operatorname{Im}\zeta \to 0$.

In the homogeneous case the domain of definition of $\mathbb{E}^{in}$ can be extended, using the following remark:

Lemma 2.29. *For all complex number a,*

$$\operatorname{Im}\zeta \in -\Gamma^\flat, \ \operatorname{Im}(a\zeta) \in -\Gamma^\flat \qquad \Rightarrow \qquad \mathbb{E}^{in}_0(a\zeta) = \mathbb{E}^{in}_0(\zeta). \tag{2.54}$$

Proof. Because $\Gamma^\flat$ is an open convex cone, one has $a \neq 0$ and $a \neq -1$. With $a_t = ta + (1-t) \neq 0$, we prove that $\mathbb{E}^{in}_0(a_t\zeta)$ is constant.

The assumptions are that $\zeta = \varpi\tilde{\xi}$ and $a\zeta = \varpi\tilde{\eta}$ with $\operatorname{Im}\tilde{\xi} \in -\Gamma$ and $\operatorname{Im}\tilde{\eta} \in -\Gamma$. Thus $\tilde{\eta} = a\tilde{\xi} + sn$, for some complex number s. For $t \in [0,1]$, $a_t\zeta = \varpi(\tilde{\xi}_t)$ with $\tilde{\xi}_t = t\tilde{\eta} + (1-t)\tilde{\xi} = a_t\tilde{\xi} + tsn$. Because

$$G_0(\xi_t) = a_t G_0(\xi) + ts\mathrm{Id},$$

the invariant spaces of $G_0(\tilde{\xi}_t)$ are those of $G(\tilde{\xi})$. Moreover, since Γ is convex, $\operatorname{Im}\tilde{\xi}_t \in -\Gamma$ and the eigenvalues of $G(\xi_t)$ do not cross the real axis. Hence $\mathbb{E}_0^{in}(\tilde{\xi}_t) = \mathbb{E}_0^{in}(\tilde{\xi})$. □

Introduce the open set

$$\mathcal{G} = \{a\zeta,\ \operatorname{Im}\zeta \in -\Gamma^\flat, a \in \mathbb{C}\backslash\{0\}\} \subset \mathbb{C} \otimes T^*\partial\Omega \approx \mathbb{C}^{1+d}/\mathbb{C}n. \tag{2.55}$$

This set is conic and stable by multiplication by complex numbers $a \neq 0$, but is *not* convex. If $a\zeta = b\zeta'$, with $\operatorname{Im}\zeta$ and $\operatorname{Im}\zeta'$ in $-\Gamma\flat$, then $\zeta' = \alpha\zeta$ with $\alpha = a/b$ and (2.64) implies that $\mathbb{E}_0^{in}(\zeta) = \mathbb{E}_0^{in}(\zeta')$. Therefore, it makes sense to extend the definition of $\mathbb{E}_0^{in}$ to the domain $\mathcal{G}$ in such a way that

$$\forall \zeta \in \mathcal{G},\ \forall a \in \mathbb{C}\backslash\{0\}: \qquad \mathbb{E}_0^{in}(a\zeta) = \mathbb{E}_0^{in}(\zeta). \tag{2.56}$$

In particular, the incoming space $\mathbb{E}^{in}(\zeta)$ is defined when $\tilde{\zeta} \in \Gamma^\flat$. We show that we can also extend the definition of $\mathbb{E}^{in}$ to this region.

Lemma 2.30. *When $\tilde{\zeta} = \varpi\tilde{\xi}$ and $\tilde{\xi} \in \Gamma$, the eigenvalues of $G_0(\tilde{\xi})$ are real and exactly N_+ are positive. The associated invariant space is $\mathbb{E}_0^{in}(\zeta)$ and has a holomorphic extension to a neighborhood of ζ.*

Moreover, there are $\varepsilon_0 > 0$ and a complex neighborhood $\mathscr{V}$ of ζ such that $\mathbb{E}^{in}$ extends holomorphical to the cone $\{\varepsilon^{-1}\zeta', \varepsilon < \varepsilon_0, \zeta' \in \mathscr{V}\}$ and

$$\forall \zeta' \in \mathscr{V}: \qquad \Pi^{in}(\varepsilon^{-1}\zeta') \to \Pi_0^{in}(\zeta') \tag{2.57}$$

One has similar results when $\theta \in -\gamma^\flat$, with $\mathbb{E}_0^{in}(-\theta)$ associated to the negative eigenvalues of $G_0(-\theta)$, so that $\mathbb{E}_0^{in}(-\theta) = \mathbb{E}^{in}(\theta)$ in accordance with (2.56).

Proof. The eigenvalues of $G_0(\tilde{\xi}) = A_n^{-1}L(\tilde{\xi})$ are the inverse of those of $L_0(\tilde{\xi})^{-1}A_n$ which are real since we assumed that $\tilde{\xi}$ is in the cone Γ. And they do not vanish since the matrices are invertible. Moreover the invariant space of $G_0(\tilde{}) = iG_0(-i\tilde{\xi})$ associated to positive eigenvalues is the invariant space of $G_0(-i\tilde{\xi})$ associated to eigenvalues in $\{\operatorname{Im}\lambda < 0\}$, that is $\mathbb{E}_0^{in}$. Thus the invariant space can be continued analytical for all small perturbations of $G_0(\tilde{\xi})$ and the remaining part of the lemma follows. □

2.10 The Lopatinski determinant(s)

We consider boundary conditions $M : \mathbb{E} \mapsto \mathbb{G}$, with with $\dim \mathbb{G} = N_+$ as above. The question under discussion is to know wether $\mathbb{E}^{in}(\zeta) \cap \ker M$ is trivial or not. There are several ways to express this condition. First, given an arbitrary scalar product in $\mathbb{E}$, one can measure the angle between $\ker M$ and $\mathbb{E}^{in}(\xi')$ through the quantity

$$D(\zeta) = \big| \det(\mathbb{H}, \mathbb{E}^{in}(\zeta)) \big| \tag{2.58}$$

where the determinant is computed by taking orthonormal bases in each space. This quantity does not depend on the choice of the bases, but it depends only on the choice of a scalar product on $\mathbb{E}$. One has

$$\mathbb{E}^{in}(\zeta) \cap \ker M = \{0\} \qquad \Leftrightarrow \qquad D(\zeta) \neq 0. \tag{2.59}$$

However, this choice ignores an important feature of the problem, which is the analytic dependence of $\mathbb{E}^{in}$. Locally in $T^*\partial\Omega - i\Gamma^\flat_\nu$, one can choose a holomorphic basis $e^{in}_k(\zeta)$ of $\mathbb{E}^{in}(\zeta)$, and form the (local) Lopatinski determinant

$$\ell(\zeta) = \det \Big[g_1, \ldots, g_{N-N_+}, e^{in}_1(\zeta), \ldots, e^{in}_{N_+}(\zeta) \Big] \tag{2.60}$$

where the g_j form a basis of $\ker M$. This function has the advantage of being holomorphic in ζ, and locally there are constants $0 < c \leq C$ such that

$$c|\ell(\zeta)| \leq D(\zeta) \leq C|\ell(\zeta)|. \tag{2.61}$$

The function ℓ can be globalized using analytic continuation and the property that $T^*\partial\Omega - i\Gamma'$ is simply connected, but the global properties of the extended function do not seem obvious.

There is an alternate way to preserve analyticity. Fix a basis e_k of $\mathbb{E}$ and for all subset $J = \{j_1, \ldots, j_{N_+}\} \subset \{1, \ldots, N\}$ of N_+ elements consider

$$\ell_J(\zeta) = \det \big[g_1, \ldots, g_{N-N_+}, \Pi^{in}(\zeta) e_{j_1}, \ldots, \Pi^{in}(\zeta) e_{N_+}. \big] \tag{2.62}$$

These functions are clearly defined and holomorphic in $T^*\partial\Omega - i\Gamma^\flat_\nu$ and

$$\mathbb{E}^{in}(\zeta) \cap \ker M \neq \{0\} \qquad \Leftrightarrow \qquad \forall J,\ \ell_J(\zeta) = 0. \tag{2.63}$$

Considering the principal part L_0 which is hyperbolic with the same cone of hyperbolic directions Γ, one can form the quantities D_0 and $\ell_{J,0}$ associated to L_0 and M. The following properties are immediate consequences of (2.56), (2.53) and Lemma 2.30.

Proposition 2.31. *i) D_0 and $\ell_{J,0}$ are defined on the set $\mathcal{G}$ and*

$$\forall \zeta \in \mathcal{G},\ \forall a \in \mathbb{C}\backslash\{0\}: \qquad D_0(a\zeta) = D_0(\zeta), \quad \ell_{J,0}(a\zeta) = \ell_{J,0}(\zeta). \tag{2.64}$$

ii) For all $\zeta \in T^*\partial\Omega - \Gamma^\flat$,

$$D(\zeta/\varepsilon) \to D_0(\zeta), \quad \ell_J(\zeta/\varepsilon) \to \ell_{J,0}(\zeta) \quad \text{as } \varepsilon \to 0. \tag{2.65}$$

iii) if $\theta \in \Gamma^\flat$, there are ε_0 and a complex neighborhood $\mathscr{V}$ of θ such that D and the ℓ_J are defined for ζ/ε if $\zeta \in \mathscr{V}$ and $\varepsilon < \varepsilon_0$ and the convergence above is true on $\mathscr{V}$.

2.11 The Lopatinski condition

First remark that if $\vartheta \in \Gamma^\flat$, then there is γ_0 such that $\gamma\vartheta \in \Gamma^\flat_\nu$ when $\gamma \geq \gamma_0$. This legitimates the following definition:

Definition 2.32. *The (weak) Lopatinski condition is satisfied in the direction $\vartheta \in \Gamma^\flat$ if and only if there is γ_0 such that $D(\zeta - i\gamma\vartheta) \neq 0$ for all $\zeta \in T^*\partial\Omega$ and $\gamma > \gamma_0$.*

Lemma 2.33. *If L satisfies the Lopatinski condition in the direction $\vartheta \in \Gamma^\flat$, then L_0 also satisfies the Lopatinski condition.*

Proof. Suppose that $D_0(\zeta) = 0$ at some $\zeta \in T^*\partial\Omega - i\gamma\vartheta$. For ε small enough, the function $g_\varepsilon(z) = D(\zeta + z\vartheta/\varepsilon)$ is defined for z in a disc centered at the origin and $g_\varepsilon \to D_0(\zeta + z\vartheta)$. Moreover, D_0 is not identically 0. Hence, by Lemma 3.5 (Hurwitz lemma if we replace D by an holomorphic local version), g_ε vanishes in a neighborhood of the origin. □

Theorem 2.34. *Suppose that the Lopatinski condition is satisfied in the direction $\vartheta \in \Gamma^\flat$. Let Σ denote the component of ϑ in $\{\zeta \in \Gamma^\flat, D_0(-i\zeta) \neq 0\}$. Then Σ is an open convex subcone of $\Gamma^\flat$ in $T^*\partial\Omega$ and the Lopatinski condition is satisfied in all direction $\theta \in \Sigma$.*

Proof. **a)** For $\zeta \in T^*\partial\Omega$, we look at the function of the complex variable z, $F_\zeta(z) = D_0(\zeta + z\vartheta)$. It is defined when $\zeta + z\vartheta \in \mathcal{G}$, in particular when $\operatorname{Im} z < 0$ since then $\zeta + z\vartheta \in T^*\partial\Omega - i\Gamma^\flat$ and, by assumption, F_ζ does not vanish there. Moreover, $-\zeta - z\vartheta \in T^*\partial\Omega - i\Gamma^\flat$ when $\operatorname{Im} z > 0$, and thus $\zeta + z\vartheta \in \mathcal{G}$. By (2.64), $F_\zeta(z) = D_0(-\zeta - z\vartheta)$ wich is $\neq 0$ by assumption. This shows that for $\zeta \in T^*\partial\Omega$, F_ζ is defined and does not vanish when $\operatorname{Im} z \neq 0$.

b) When $\theta \in \Sigma$, $\mathrm{Im}\,(-i(\theta+z\vartheta)) = -\theta - \mathrm{Re}\, z\vartheta \in -\Gamma^\flat$ when $\mathrm{Re}\, z \geq 0$ thus $-i(\theta+z\vartheta) \in T^*\partial\Omega - i\Gamma^\flat$ and $\theta + z\nu \in \mathcal{G}$. Thus, F_θ is defined for $\mathrm{Re}\, z \geq 0$. It does not vanish when $\mathrm{Im}\, z \neq 0$ by step a), and it does not vanish when $z = 0$ since $F_\theta(0) = D_0(\theta) = D_0(-i\theta)$ which is $\neq 0$ by assumption. Therefore, $F_\theta(z) \neq 0$ when $\mathrm{Re}\, z = 0$.

Moreover, for $|z|$ large in $\mathrm{Re}\, z \geq 0$, one has $\vartheta + z^{-1}\zeta \in \Gamma^\flat \subset \mathcal{G}$ and $F_\zeta(z) = D_0(\vartheta + z^{-1}\zeta) = D_0(-i(\vartheta + z^{-1}\zeta)) \neq 0$ since $D_0(\vartheta \neq 0)$.

This shows that F_θ does not vanish when $\mathrm{Re}\, z = 0$ or when $\mathrm{Re}\, z \geq 0$ and $lz|$ is large. Since $F_\vartheta(z) = D_0((1+z)\vartheta = D_0(\vartheta) \neq 0$ for all z such that $\mathrm{Re}\, z \geq 0$, Lemma 3.6 by deformation that F_θ does not vanish either on the domain $\{\mathrm{Re}\, z \geq 0\}$:

$$\forall \theta \in \Sigma,\ \forall z,\ \mathrm{Re}\, z \geq 0 \quad \Rightarrow \quad D_0(\theta + z\vartheta) \neq 0. \tag{2.66}$$

Because $\mathrm{Re}\, 1/z \geq 0$ when $\mathrm{Re}\, z \geq 0$, the homogeneity of D_0, implies that $D_0(\vartheta + z\theta) \neq 0$ when $\mathrm{Re}\, z \geq 0$ and $z \neq 0$. This property is also true at $z = 0$, and hence

$$\forall \theta \in \Sigma,\ \forall z,\ \mathrm{Re}\, z \geq 0 \quad \Rightarrow \quad D_0(\vartheta + z\theta) \neq 0. \tag{2.67}$$

In particular, this applies to z real nonnegative, and by homogeneity, one has $D_0(t\theta' + s\nu') \neq 0$ when $t > 0$ and $s \geq 0$. This extends to $t = 0$. Thus the segment $[\nu, \theta']$ is contained in Σ and Σ is star shaped with respect to ν.

c) Let $\zeta \in T^*\partial\Omega$ and $\theta \in \Sigma$. For $\gamma > \gamma_0$, we look at the function of z, $G_\gamma(z) = D(\zeta - i\gamma\vartheta - iz\theta)$, which is defined for $\mathrm{Re}\, z \geq 0$ since then $\mathrm{Im}\,(\zeta - i\gamma\theta^\flat - iz\theta) = -\gamma\theta^\flat - \mathrm{Re}\, z\theta \in -\Gamma^\flat_\nu - \Gamma^\flat \subset -\Gamma^\flat_\nu$. It does not vanish when $\mathrm{Re}\, z = 0$, since the Lopatinski condition is satisfied in the direction ϑ.

Moreover, when z is large, setting $\hat{z} = z/|z|$, one has

$$G_\gamma(z) = D\big(-i\hat{z}\theta + |z|^{-1}(\zeta - i\gamma\vartheta)\big)$$

By iii) of Proposition 2.31, since $\theta \in \Gamma^\flat$, this converges to $D_0(-i\hat{z}\theta) = D_0(-i\theta) \neq 0$ if $\mathrm{Re}\, \hat{z} \geq 0$. This implies that G_γ does not vanish in the half space $\mathrm{Re}\, z \geq 0$, either when $\mathrm{Re}\, z = 0$ or when $|z| \geq R_0(1+\gamma)$, for some R_0 large enough.

Therefore, applying Lemma 3.6, to prove that

$$\forall \zeta \in T^*\partial\Omega,\ \forall \gamma > \gamma_0, \forall z,\ \mathrm{Re}\, z \geq 0 \quad \Rightarrow \quad D(\zeta - i\gamma\vartheta - iz\theta) \neq 0. \tag{2.68}$$

it is sufficient to show that for γ_1 large

$$\gamma \geq \gamma_1,\ |z| \leq R_0(1+\gamma) : \qquad D(\zeta - i\gamma\vartheta - iz\theta) \neq 0. \tag{2.69}$$

Here we factor out γ and use again the Proposition 2.31 which implies that

$$G_\gamma(z) = D(\gamma(-i\vartheta - i\hat{z}\theta + \gamma^{-1}\zeta) \to D_0(-i(\vartheta + \hat{z}\theta)),$$

where $\hat{z} = z/\gamma$ is bounded. By step (2.67) the limit does not vanish and is bounded from below since $|\hat{z}|$ is bounded. Therefore, (2.69) and (2.68) follow.

d) Because Σ is open, one can replace θ by $\theta - \delta\vartheta$ for some $\delta > 0$ small, and (2.68) implies that

$$\forall \zeta \in T^*\partial\Omega, \ ,\forall z, \ \operatorname{Re} z > \beta \quad \Rightarrow \quad D(\zeta - iz\theta) \neq 0. \tag{2.70}$$

This shows that the Lopatinski condition is satisfied in the direction θ'.

Applying step a), this implies that Σ is star shaped with respect to θ' and the proof of the theorem is complete. □

Theorem 2.35. *If M satisfies the uniform Lopatinski condition in a direction $\vartheta \in \Gamma^\flat$, then $\Sigma = \Gamma^\flat$ and the uniform Lopatinski condition is satisfied in all directions $\theta \in \Gamma^\flat$.*

Proof. We have seen that $\Gamma^\flat - i\Gamma^\flat \in \mathcal{C}$ and that Δ is continuous there. The uniform Lopatinski condition implies that $|\Delta(a\theta)| \geq c$ when $\theta \in \Gamma^\flat \operatorname{Im} a < 0$. Hence by continuity $|\Delta(\theta)| \geq c$, implying that $\theta \in \Sigma$.

By Proposition 2.22, there is $\varepsilon > 0$ such that M' satisfies the Lopatinski condition in the direction ϑ if $|M - M'| \leq \varepsilon$, and thus in all direction $\theta \in \Sigma = \Gamma^\flat$ by Theorem 2.34 and the remark above. By Proposition 2.22, this implies that the uniform Lopatinski condition is satisfied in all directions $\theta \in \Gamma^\flat$. □

3 Appendix

3.1 Laplace Fourier Transform

If $u \in \mathscr{D}'(\mathbb{R}^d)$, let M denote the set of $\eta \in \mathbb{R}^d$ such that $e^{\eta \cdot x} u \in \mathscr{S}'(\mathbb{R}^d)$.

Lemma 3.1 ([Hör] Lemma 7.4.1). *M is convex.*

Proof. Note that if $\psi \in C^\infty(\mathbb{R}^d)$ is bounded as well as its derivatives at all order, then the mapping $\varphi \mapsto \psi\varphi$ is continuous in $\mathscr{S}$, and therefore $u \mapsto \psi u$ is a continuous map in $\mathcal{S}'$.

It $\eta_1 \in M$ and $\eta_2 \in M$, for $t \in [0, 1]$ and $\eta = t\eta_1 + (1-t)\eta_2$, one has $e^{\eta \cdot x} = \psi(e^{\eta_1 \cdot x} + e^{\eta_2 \cdot x})$ where ψ is bounded and has bounded derivatives, implying that $\eta \in M$. □

Lemma 3.2. *If the interior M° of M is not empty, then there is an holomorphic function U on $\mathbb{R}^d + iM^\circ$ such that for $\eta \in M^\circ$, let $U(\cdot + i\eta)$ is the Fourier transform of $e^{\eta\cdot x}u$ in $\mathscr{S}'(\mathbb{R}^d)$.*

Proof. Let $\in M^\circ$. For $\varepsilon > 0$ small enough, the points $\underline{\eta} \pm \varepsilon e_j$ belong to M, where $\{e_j\}$ denote a basis of $\mathbb{R}^d$. Denote by η_k the set of such points. Then, for $|\eta - \underline{\eta}|$ small enough, the function

$$\psi_\eta = \big(\sum e^{(\eta_k - \eta)\cdot x} \big)^{-1}$$

in in the Schwartz class $\mathscr{S}$, and is bounded in this space. This implies that the Fourier transform $\hat{u}_\eta$ of $u_\eta = e^{\eta\cdot x}u$ is C^∞ in ξ and in η, for η close to $\underline{\eta}$. Let $U(\xi + i\eta) = \hat{u}_\eta(\xi)$.

Moreover, both $i\partial_{\xi_j}\hat{u}_\eta$ and $\partial_{\eta_j}\hat{u}_\eta$ are the Fourier transform of $x_j u_\eta$. Hence there are equal, implying that the Cauchy Riemann equations $(\partial_{\xi_j} + i\partial_{\eta_j})U = 0$ are satisfied and U is holomorphic in $\xi + i\eta$. □

Theorem 3.3. *Let Γ be a convex open cone in $\mathbb{R}^d$. If $U(\xi)$ is an holomorphic function on $\mathscr{U} := \{\xi \in \mathbb{R}^d + i\Gamma,\ |\mathrm{Im}\,\xi| > \gamma_0\}$ and satisfies there*

$$|U(\xi)| \le C(1 + |\xi|)^m \tag{3.1}$$

then U is the Fourier Laplace transform of a distribution supported in

$$\hat{\Gamma} = \{x : \forall \xi,\ \xi \cdot x \le 0\}. \tag{3.2}$$

Proof. For $\eta \in \Gamma$ with $|\eta| > \gamma_0$, the function $U(\cdot + i\eta)$ is slowly growing at infinity and is the Fourier transform of $u_\eta \in \mathscr{S}'(\mathbb{R}^d)$. Moreover, the Cauchy Riemann equation implies that $\partial_{\eta_j}u_\eta = x_j\hat{u}_\eta$, hence that $u = e^{\eta\cdot x}u_\eta$ is independent of η.

The estimates imply that the u_η are $O(|\eta|^m)$ in $\mathscr{S}'$, hence . □

3.2 Proof of Lemma 2.12

Proposition 3.4. *The set $\mathcal{P} = \{(\zeta, \Pi^{in}(\zeta)); \mathrm{Im}\,\zeta < 0\}$ is (real) semi-algebraic, that is a finite union of finite intersections of sets defined by polynomial equations or inequalities.*

Proof. The characteristic polynomial $p(z,\zeta) = \det(z\mathrm{Id} - G(\zeta)$ can be factored as $p = p_+p_-$ where $p_+(\cdot,\zeta)$ [resp. $p_-(\cdot,\zeta)$] has all its roots in $\mathrm{Im}\,z > 0$ [resp. $\mathrm{Im}\,z < 0$]. There are polynomials in z, with analytic coefficients in ζ, denoted by $u_\pm(z,\zeta)$ such that $p_+u_+ + p_-u_- = 1$, wich are uniquely determined if one adds the condition $\deg u_+ < \deg p_- := N_-$, $\deg u_- < \deg p_+ := N_+$. Note that $N_\pm$ are fixed. The projector $\Pi^{in}(\zeta)$ is

$$\Pi^{in}(\zeta) = (u_- p_-)(A(\zeta), \zeta). \tag{3.3}$$

We consider the set $\tilde{\mathcal{P}}$ of $\zeta = (\tau, \xi') \in \mathbb{C} \times \mathbb{R}^{d-1}$, $(a_1, \dots, a_{N_-}) \in \mathbb{C}^{N_-}$, $(b_1, \dots, b_{N_+}) \in \mathbb{C}^{N_+}$ $(u_1, \dots, u_{N_-}) \in \mathbb{C}^{N_-}$, $(v_1, \dots, v_{N_+}) \in \mathbb{C}^{N_-}$ and matrices Π satisfying the conditions:

$$\operatorname{Im} \tau < -\gamma_0, \quad \operatorname{Im} a_j < 0, \quad \operatorname{Im} b_j > 0, \tag{3.4}$$

$$\big(\sum v_j z^{j-1}\big) \prod (z - a_j) + \big(\sum u_j z^{j-1}\big) \prod (z - b_j) = 1 \tag{3.5}$$

$$\prod (z - a_j) \prod (z - b_j) = \det(z - G(\zeta)), \tag{3.6}$$

$$\Pi = \big(\sum v_j G(\zeta)^{j-1}\big) \prod (G(\zeta) - a_j \mathrm{Id}) \tag{3.7}$$

The second and third conditions are polynomial conditions on the (a_j, b_j, u_j, v_j) and ζ. Thus $\tilde{\mathcal{P}}$ is semi-algebraic. Now, $\mathcal{P}$ is just the projection of $\tilde{\mathcal{P}}$ in the space of (ζ, Π), therefore is semi-algebraic by Tarski-Seidenberg Theorem. □

Proof of Lemma 2.12. Consider a basis $\{e_k\}$ of $\mathbb{E}$. If Π is a $N \times N$ matrix, for $I \subset \{1, \dots, N\}$ with $|I| = \dim \mathbb{E}$, we can form the matrix $[M\Pi]_I$ with columns $M\Pi e_k$ for $k \in I$ and define

$$\ell(\Pi) = \sum_I |\det([M\Pi]_I)|^2. \tag{3.8}$$

The Lopatinski condition is that

$$\ell(\Pi^{in}(\zeta)) > 0 \qquad \text{when } \operatorname{Im} \tau < -\gamma_0. \tag{3.9}$$

Consider the set $\mathcal{Q}$ of (t, δ, η, Π) such that

$$\operatorname{Im} \tau \le -\gamma - 1, \ |\zeta| \le t, \ (\zeta, \Pi) \in \mathcal{P}, \ \delta = \ell(\Pi). \tag{3.10}$$

This set is semi-algebraic and therefore the function

$$f(t) = \inf\{\delta; \exists (\zeta, \Pi) \ : \ (t, \delta, \zeta, \Pi) \in \mathcal{Q}\} \tag{3.11}$$

is semi-algebraic by Corollary A.2.4 in [Hör]. The Lopatinski condition implies that $f(t) > 0$ for all t. Hence, by Theorem A.2.5 in [Hör], there are a rational number α and $c > 0$ such that

$$f(t) = ct^\alpha (1 + o(t)), \qquad t \to +\infty. \tag{3.12}$$

This implies that for $|\zeta|$ large enough and $\operatorname{Im}\tau \leq -\gamma - 1$, one has

$$\ell(\Pi^{in}(\zeta) \geq \frac{1}{2} c \langle\zeta\rangle^{\alpha}. \tag{3.13}$$

The Lopatinski condition implies that this estimate is also valid (possibly with another constant $c > 0$) on any compact domain in $|\zeta|$. Hence is is satisfied for all ζ such that $\operatorname{Im}\tau \leq -\gamma - 1$.

In particular, there is another constant $c > 0$ such that for all ζ there is I satisfying

$$|\ell_I(\Pi^{in}(\zeta)|^2 \geq c' \langle\zeta\rangle^{\alpha}. \tag{3.14}$$

Let $u \in \mathbb{E}^{in}(\zeta)$. The Lopatinski condition implies that u is uniquely determined by Mu and

$$u = \sum_{j \in I} a_j \Pi^{in}(\zeta) e_j \tag{3.15}$$

where $a = (a_j)_{j \in I}$ solves $[M\Pi^{in}]_I a = g$. Because Π^{in} has polynomial bounds in $|\zeta|$, the estimate (3.14) implies that for some C and m:

$$|a| \leq C\langle\zeta\rangle^m |g|$$

The estimate (2.23) follows. □

3.3 The analogue of Rouché's theorem

Lemma 3.5. *Suppose that D_n is a sequence of functions on $\dot{H} = \{\operatorname{Re} z >\}$, which converge uniformly to D on compact subsets of $\dot{H}$. Suppose that for all $\underline{z} \in \dot{H}$ there is a neighborhood ω of $\underline{z}$, a sequence of holomorphic functions ℓ_n on ω for $n \geq n_0$, which converge to ℓ, and a constant $C > 1$ such that*

$$\forall z \in \omega, \forall n \geq n_0, \quad \frac{1}{C}|\ell_n(z)| \leq D_n(z) \leq C|\ell_n(z)| \tag{3.16}$$

and $\ell_n \to \ell$ Suppose that D is not identically zero. Then, if D vanishes at $z_0 \in \dot{H}$, there is a sequence $z_n \to z$ such that $D_n(z_n) = 0$.

Proof. **a)** From the lemma above, we know that $D(\cdot)$ cannot vanish identically on any open set since it does not vanish at infinity $\dot{H}$.

b) If $D(z) = 0$, then by assumption there are holomorphic functions $\ell_n \to \ell$ on a neighborhood ω such that the zeros of D_n [resp. D] in ω are the zeros of the ℓ_n. Since ℓ is not identically zero, z is a zero of finite order m and on a possibly smaller neighborhood of z, for n large enough, ℓ_n has the m zeros, counted with their multiplicities. □

Lemma 3.6. *Suppose that D is a continuous function on $\mathcal{H} := [0,1] \times H$ where $H = \{z \in \mathbb{C}, \operatorname{Re} z \geq 0\}$. Suppose that for all $(t_0, z_0) \in \mathcal{H}$, there is a neighborhood of (t_0, z_0), a function ℓ on this neighborhood, continuous in t and holomorphic in z, and a constant $C > 1$ such that*

$$\frac{1}{C}|\ell(t,z)| \leq D(t,z) \leq C|\ell(t,z)|. \tag{3.17}$$

Suppose that there is $R > 0$ such that for all $t \in [0,1]$, $D(t,z) \neq 0$ when $\operatorname{Re} z = 0$ and when $|z| \geq R$. Suppose that $D(0,z) \neq 0$ for all $z \in H$. Then if $D(1,\cdot)$ does not vanish on H.

Proof. a) We show that $D(t,\cdot)$ cannot vanish identically on any open set. If it would, let Z denote the non empty set of points $z \in \dot{H}$ such that $D(t,\cdot)$ vanishes identically on a neighborhood of z. It is open by definition. If z_n is a sequence of points in Z which converge to $z \in \dot{H}$, the assumption implies that on a neighborhood ω of z, the zeros of D are zeros of an holomorphic function ℓ. In particular, for n large $z_n \in \omega$ and $\ell(z_n) = 0$. Therefore, the zeros of ℓ have an accumulation at point, implying that ℓ and therefore D must vanish identically on ω. Therefore Z is open and closed and $Z = \dot{H}$, which contradicts the assumption that $D(t,\cdot)$ does not vanish at infinity. .

b) The set N of (t,z) such that $D(t,z) = 0$ is compact in $]0,1] \times \dot{H}$ where $\dot{H} = \{\operatorname{Re} z > 0\}$ is the interior of H. If it is not empty, let $t_0 = \min\{t, (t,z) \in N\}$ and let $z_0 \in \dot{H}$ such that $D(t_0, z_0) = 0$. Then $t_0 > 0$.

Let ℓ be a function satisfying (3.17) on a neighborhood of (t_0, z_0). By a), $\ell(t_0,\cdot)$ it is not identically 0, and therefore it is has a zero of finite order at z_0 and therefore does not vanish on the boundary of a small disc containing z_0. Hence, by Rouché's theorem, $\ell(t,\cdot)$ has a root in this disc for $t - t_0$ small, which contradicts the definition of t_0. □

References

[BeSe] S.Benzoni-Gavage, D.Serre, *Multidimensional hyperbolic partial differential equations. First-order systems and applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

[ChPi] J.Chazarain, A.Piriou *Caractérisation des problèmes mixtes hyperboliques bien posés*, Ann. Inst. Fourier, 22 (1972) pp 193–237

[Fr1] K.0. Friedrichs, *Symmetric hyperbolic linear differential equations,* Comm. Pure Appl.Math., 7 (1954), pp 345-392.

[Fr2] K.0. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure Appl.Math., 11 (1958), pp 333-418.

[Gar] L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. 85, (1951) pp 1–62

[GMWZ] O.Gu'es, G.Métivier, M.Williams, K.Zumbrun, *Uniform stability estimates for constant-coefficient symmetric hyperbolic boundary value problems,* Comm. in Partial Diff. Equ., 32 (2007) pp 579-590

[Her] R. Hersh, *Mixed problems in several variables*, J Math. Mech. 12 (1963) pp 317-334.

[Hör] L.Hörmander, *The Analysis of Partial Differential Operators*, Springer Verlag

[Ka] K.Kasahara, *On weak well-posedness of mixed problems for hyperbolic systems*, Publ. Res. Inst. Math. Sci. 6 (1970/71), 503?514.

[Kr] H. Kreiss, *Initila Boundary Value Problems for Hyperbolic Systmes*, Comm. on Pure and Appl. Math., 23 (1970), pp 277-298.

[Maj] A.Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc. 41 (1983), no. 275,

[Me1] G.Métivier, *L^2 well posed Cauchy Problems and Symmetrizability*, J. Ecole Polytechnique, 1 (2014) pp 39–70

[Me2] *On the L^2 well posedenss of Hypebolic Initial Boundary Value Problems*, preprint.

[Me3] G.Métivier, *Small viscosity and boundary layer methods. Theory, stability analysis, and applications*, Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2004.

[MZ] G.Métivier, K.Zumbrum, *Hyperbolic Boundary Value Problems for Symmetric Systems with Variable Multiplicities*, J. Diff. Equ., 211 (2005) 61-134.