Scalar conservation laws

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1 Linear Solution Operator

We consider the following *scalar viscous conservation law*

$$u_t + f(u)_x = u_{xx}, \tag{1.1}$$

where $u \in C^2(\mathbb{R}^2 \to \mathbb{R}), f \in C^2(\mathbb{R} \to \mathbb{R})$, $u = u(x,t)$.
Let us consider the stability of the standing wave solution

$$\bar{u} = \bar{u}(x), \lim_{x\to\pm\infty} \bar{u}(x) = \bar{u}_\pm \tag{1.2}$$

satisfying the Lax condition

$$(\mathcal{H}) \quad f'(\bar{u}_+) < 0 < f'(\bar{u}_-)$$

Thus obviously $\bar{u}$ satisfies

$$f(\bar{u}(x))_x = \bar{u}_{xx} \tag{1.3}$$

or equivalently

$$f'(\bar{u})\bar{u}_x = \bar{u}_{xx} \tag{1.4}$$

To linearize the equation (1.1), we substitute $u = v + \bar{u}$ in (1.1) to get

$$\begin{aligned}
(v+\bar{u})_t + (f(v+\bar{u}))_x &= (v+\bar{u})_{xx} \\
v_t + (f'(v+\bar{u}))(v_x+\bar{u}_x) &= v_{xx} + \bar{u}_{xx} \\
v_t + [f'(\bar{u}) + f''(\bar{u})v + \frac{1}{2}f'''(\bar{u})v^2 + o(v^2)](v_x+\bar{u}_x) &= v_{xx} + \bar{u}_{xx}
\end{aligned}$$

Expand the left hand side of the last equality, and throw away nonlinear terms about v, to get

$$v_t + f'(\bar{u})v_x + f''(\bar{u})\bar{u}_x v + f'(\bar{u})\bar{u}_x = v_{xx} + \bar{u}_{xx}$$

Utilizing (1.4) to simply the above equation to

$$v_t + f'(\bar{u})v_x + f''(\bar{u})\bar{u}_x v = v_{xx}$$

We get the the linearized equation about solution $\bar{u}$

$$v_t - Lv := v_t + (f'(\bar{u})v)_x - v_{xx} = 0 \tag{1.5}$$

where $Lv := v_{xx} - (f'(\bar{u})v)_x$.

Letting $\tilde{u}$ be a second solution of (1.1), and defining perturbation $u := \tilde{u} - \bar{u}$, then $\tilde{u} = u + \bar{u}$, substitute this into (1.1),

$$\begin{aligned}
u_t + f'(u+\bar{u})(u_x + \bar{u}_x) &= u_{xx} + \bar{u}_{xx} \\
u_t - Lu &= u_{xx} + \bar{u}_{xx} - f'(u+\bar{u})(u_x + \bar{u}_x) - Lu \\
&= u_{xx} + \bar{u}_{xx} - f'(u+\bar{u})(u_x + \bar{u}_x) - u_{xx} + (f'(\bar{u})u)_x \\
&= \bar{u}_{xx} - (f(u+\bar{u}))_x + (f'(\bar{u})u)_x \\
&= [\bar{u}_x - f(u+\bar{u}) + f'(\bar{u})u]_x \\
&= (N_{\bar{u}}(u))_x
\end{aligned}$$

where $N_{\bar{u}}(u) := \bar{u}_x - f(u+\bar{u}) + f'(\bar{u})u$. So we have the *perturbation equation*

$$u_t - Lu = (N_{\bar{u}}(u))_x \tag{1.6}$$

The homogeneous linearized equation

$$v_t - Lv := v_t + (f'(\bar{u})v)_x - v_{xx} = 0, v(x,0) = g(x) \tag{1.7}$$

2 The Asymptotic Eigenvalue Equations

The eigenvalue equation $Lw = \lambda w$ associated with (1.7) is,

$$w'' - (f'(\bar{u})w)' = \lambda w. \tag{2.1}$$

Written as first-order system in the variable $W = (w, w')^t$, this becomes

$$W' = \mathbb{A}(x;\lambda)W, \tag{2.2}$$

where

$$\mathbb{A}(x;\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda + f''(\bar{u})\bar{u}_x & f'(\bar{u}) \end{pmatrix} \tag{2.3}$$

We begin by studying the limiting, constant coefficient systems $L_{\pm}w = \lambda w$ of (2.1) at $\pm\infty$,

$$w'' - (f'(\bar{u}_{\pm})w)' = \lambda w. \tag{2.4}$$

Or, written as a first-order system,

$$W' = \mathbb{A}_{\pm}(\lambda)W, \tag{2.5}$$

where

$$\mathbb{A}_{\pm}(\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda & f'(\bar{u}_{\pm}) \end{pmatrix} \tag{2.6}$$

since $\bar{u}'_{\pm} = 0$.
The normal modes of (2.5) are $V_j^{\pm} e^{\mu_j^{\pm} x}$, $j = 1, 2$, where $\mu_j^{\pm}, V_j^{\pm}$ are the eigenvalues and eigenvectors of $\mathbb{A}_{\pm}$; they are easily seen to satisfy

$$V_j^{\pm} = \begin{pmatrix} v_j^{\pm} \\ \mu_j^{\pm} v_j^{\pm} \end{pmatrix}, v_j^{\pm} \in \mathbb{C}, \tag{2.7}$$

and

$$(\mu_j^{\pm})^2 - f'(\bar{u}_{\pm})\mu_j^{\pm} - \lambda = 0 \tag{2.8}$$

Then we can solve for $\mu_j^{\pm}$ as

$$\mu_j^{\pm}(\lambda) = \frac{f'(\bar{u}_{\pm}) \pm \sqrt{(f'(\bar{u}_{\pm}))^2 + 4\lambda}}{2} \tag{2.9}$$

Lemma 2.1. *Let*

$$\mu_1(\lambda) = \frac{a - \sqrt{a^2 + 4\lambda}}{2}$$

and

$$\mu_2(\lambda) = \frac{a + \sqrt{a^2 + 4\lambda}}{2}$$

be two solutions to the equation $\mu^2 - a\mu - \lambda = 0$, *where* $a \in \mathbb{R}, \lambda \in \mathbb{C}$. *Then* $\mathrm{Re}\mu_1(\lambda) < 0 < \mathrm{Re}\mu_2(\lambda)$ *or* $\mathrm{Re}\mu_1(\lambda) > 0 > \mathrm{Re}\mu_2(\lambda)$ *(i.e. real parts of two solutions have different signs) if and only if* $\lambda \in \Lambda$, *where*

$$\Lambda = \left\{\lambda \in \mathbb{C} : a^2\mathrm{Re}(\lambda) + (\mathrm{Im}(\lambda))^2 > 0\right\}$$

Proof. First we have the following two relations

$$\begin{aligned}\mu_1+\mu_2 &= a\\ \mu_1\mu_2 &= -\lambda\end{aligned}$$

We set $\mu_1 = x_1 + iy_1, \mu_2 = x_2 + iy_2$,for $x_j, y_j \in \mathbb{R}, j = 1, 2$, then we get

$$x_1 + x_2 = \mathrm{Re}(\mu_1) + \mathrm{Re}(\mu_2) = a$$
$$y_1 + y_2 = \mathrm{Im}(\mu_1) + \mathrm{Im}(\mu_2) = 0$$

Note that

$$\begin{aligned}-\lambda &= \mu_1\mu_2 = (x_1+iy_1)(x_2+iy_2)\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\\ &= (x_1x_2 + y_1^2) + i(-x_1y_1 + x_2y_1)\\ &= (x_1(a-x_1) + y_1^2) + iy_1(x_2 - x_1)\\ &= (x_1(a-x_1) + y_1^2) + iy_1(a - 2x_1)\end{aligned}$$

Thus we have

$$\mathrm{Re}(\lambda) = -(x_1(a-x_1) + y_1^2) = x_1(x_1-a) - y_1^2 \tag{2.10}$$
$$\mathrm{Im}(\lambda) = -y_1(a-2x_1) = y_1(2x_1 - a) \tag{2.11}$$

The situation that real parts of solutions μ_1, μ_2 have different signs is equivalent to the condition

$$x_1x_2 = x_1(a - x_1) < 0$$

which is again equivalent to

$$x_1 < 0 \text{ or } x_1 > a, \text{ if } a \geq 0; \tag{2.12}$$
$$x_1 < a \text{ or } x_1 > 0, \text{ if } a < 0. \tag{2.13}$$

Now we multiply both sides of (2.10) by $(2x_1 - a)^2$ and substitute (2.11) into the result to get

$$(2x_1-a)^2\mathrm{Re}(\lambda) = x_1(x_1-a)(2x_1-a)^2 - y_1^2(2x_1-a)^2 \tag{2.14}$$
$$= x_1(x_1-a)(2x_1-a)^2 - (\mathrm{Im}(\lambda))^2 \tag{2.15}$$

Now we are trying to find a relation which is satisfied by $\mathrm{Re}(\lambda), \mathrm{Im}(\lambda)$ subject to conditions (2.12) and (2.13). Hence we make a change of variable $x_1 - \frac{a}{2} = \zeta$, then $x_1 = \zeta + \frac{a}{2}, x_1 - a = \zeta - \frac{a}{2}$. And ζ satisfies the condition

$$\zeta < -\frac{a}{2} \text{ or } \zeta > \frac{a}{2}, \text{ if } a \geq 0;$$
$$\zeta < \frac{a}{2} \text{ or } \zeta > -\frac{a}{2}, \text{ if } a < 0.$$

or in short just

$$|\zeta| > \left|\frac{a}{2}\right|.$$

Equation (2.15) becomes

$$\begin{aligned}
(2\zeta)^2\mathrm{Re}(\lambda) &= (\zeta + \frac{a}{2})(\zeta - \frac{a}{2})(2\zeta)^2 - (\mathrm{Im}(\lambda))^2 \\
4\zeta^2\mathrm{Re}(\lambda) &= 4\zeta^4 - a^2\zeta^2 - (\mathrm{Im}(\lambda))^2 \\
0 &= 4\zeta^4 - (a^2 + 4\mathrm{Re}(\lambda))\zeta^2 - (\mathrm{Im}(\lambda))^2
\end{aligned}$$

View the above equation as a quadratic equation about ζ^2, the larger root of this equation is given by

$$\frac{a^2 + 4\mathrm{Re}(\lambda) + \sqrt{(a^2 + 4\mathrm{Re}(\lambda))^2 + 16(\mathrm{Im}(\lambda))^2}}{8} > \frac{a^2}{4}$$

and this root has to be greater than $\frac{a^2}{4}$ because $\zeta^2 > \frac{a^2}{4}$. This inequality simplifies to read

$$\sqrt{\left(\frac{a^2}{4} + \mathrm{Re}(\lambda)\right)^2 + (\mathrm{Im}(\lambda))^2} \;>\; \frac{a^2}{4} - \mathrm{Re}(\lambda)$$

and then

$$a^2\mathrm{Re}(\lambda) + (\mathrm{Im}(\lambda))^2 \;>\; 0.$$

□

From this lemma we know that the *region of consistent splitting* Λ of system (2.5) is intersection of the following two sets

$$\Lambda_\pm = \left\{\lambda \in \mathbb{C} : (f'(\bar{u}_\pm))^2\mathrm{Re}(\lambda) + (\mathrm{Im}(\lambda))^2 > 0\right\} \tag{2.16}$$

then the Lax condition

$$f'(\bar{u}_+) < 0 < f'(\bar{u}_-) \tag{2.17}$$

and Lemma 2.1 gives the two eigenvalues of $\mathbb{A}_\pm$ as following,

$$\mathrm{Re}\mu_1(\lambda) < 0 < \mathrm{Re}\mu_2(\lambda). \tag{2.18}$$

3 Asymptotic Behavior of the Stationary Solution $\bar{u}$

The stationary wave solution $\bar{u}(x)$ satisfies (1.3):

$$f(\bar{u}(x))_x = \bar{u}_{xx}$$

and the asymptotic conditions

$$\lim_{x\to+\infty} \bar{u}(x) = \bar{u}_+; \tag{3.1}$$

$$\lim_{x\to-\infty} \bar{u}(x) = \bar{u}_-; \tag{3.2}$$

$$\lim_{x\to+\infty} \bar{u}'(x) = 0; \tag{3.3}$$

$$\lim_{x\to-\infty} \bar{u}'(x) = 0. \tag{3.4}$$

We integrate (1.3) from $x(>0)$ to $+\infty$, to get

$$\bar{u}_x = f(\bar{u}) - f(\bar{u}_+) \tag{3.5}$$

Rewrite as

$$\bar{u}_x = F(\bar{u}) := f(\bar{u}) - f(\bar{u}_+) \tag{3.6}$$

We notice that $\bar{u} = \bar{u}_+$ is a critical point of the ODE (3.6), thus for $\bar{u}$ to be a stable solution of ODE (3.6), it is necessary that $F'(\bar{u}_+) \leq 0$, i.e., $F'(\bar{u}_+) = f'(\bar{u}_+) \leq 0$. Let $\phi = \bar{u}$, then (3.6) reads

$$\phi_x = F(\phi) := f(\phi) - f(\bar{u}_+) \tag{3.7}$$

Lemma 3.1. *Consider the initial value problem* (3.7) *with* $\phi(x_0) = \bar{u}_0$. *If we assume that* $f'(\bar{u}_+) < 0$, *then there are positive constants* $\delta > 0$ *and* $\alpha > 0$ *that are independent of the choice of the initial time* x_0 *such that the solution* $x \mapsto \phi(x)$ *of the initial value problem* (3.7) *satisfies*

$$|\phi(x) - \bar{u}_+| \leq |\phi(x_0) - \bar{u}_+| \, e^{-\alpha(x-x_0)} \tag{3.8}$$

for $x \geq x_0$ *whenever* $|\phi(x_0) - \bar{u}_+| \leq \delta$.

Proof. Pick any $\delta_0 > 0$, then $u \mapsto f''(u)$ is a continuous function on the interval $[\bar{u}_+ - \delta_0, \bar{u}_+ + \delta_0]$. Thus we let

$$C := \sup_{u\in[\bar{u}_+-\delta_0,\bar{u}_++\delta_0]} |f''(u)| \tag{3.9}$$

Then we choose $\delta > 0$ small enough such that $f'(\bar{u}_+) + \delta C < 0, \delta < \delta_0$, and define $\alpha := -f'(\bar{u}_+) - \delta C > 0$.
If $|\phi(x_0) - \bar{u}_+| \leq \delta$, then there is a maximal half-open interval

$$\mathfrak{J} := \{x \in \mathbb{R} : x_0 \leq x < \xi\}$$

such that the solution $x \mapsto \phi(x)$ of the differential equation with initial condition $\phi(x_0) = \bar{u}_0$ exists and satisfies the inequality

$$|\phi(x) - \bar{u}_+| < \delta \tag{3.10}$$

on the interval $\mathfrak{J}$.
We rewrite the equation (3.7) as

$$\begin{aligned}
(\phi - \bar{u}_+)_x &= f(\phi) - f(\bar{u}_+) \\
&= f'(\bar{u}_+)\phi + f(\phi) - f'(\bar{u}_+)\phi - f(\bar{u}_+) \\
&= f'(\bar{u}_+)(\phi - \bar{u}_+) + f(\phi) - f(\bar{u}_+) - f'(\bar{u}_+)(\phi - \bar{u}_+)
\end{aligned}$$

According to Variation of Constants Formula, we have the following

$$\begin{aligned}
\phi(x) - \bar{u}_+ &= e^{f'(\bar{u}_+)(x-x_0)}(\phi(x_0) - \bar{u}_+) \\
&\quad + \int_{x_0}^{x} e^{f'(\bar{u}_+)(x-y)} \left[f(\phi(y)) - f(\bar{u}_+) - f'(\bar{u}_+)(\phi(y) - \bar{u}_+)\right] dy
\end{aligned}$$

for all $x \in \mathfrak{J}$.
Notice that because $x_0 \leq y \leq x < \xi$, $y \in \mathfrak{J}$, so

$$|\phi(y) - \bar{u}_+| < \delta \tag{3.11}$$

By Taylor's Theorem, we have

$$f(\phi(y)) - f(\bar{u}_+) - f'(\bar{u}_+)(\phi(y) - \bar{u}_+) = \frac{1}{2}f''((1-t)\phi(y) + t\bar{u}_+)(\phi(y) - \bar{u}_+)^2$$

for some $0 \leq t \leq 1$.
Then $(1-t)\phi(y) + t\bar{u}_+ \in [\bar{u}_+ - \delta_0, \bar{u}_+ + \delta_0]$ implies

$$|f''((1-t)\phi(y) + t\bar{u}_+)| \leq C \tag{3.12}$$

Utilize (3.11) and (3.12), we have the estimate

$$\begin{aligned}|\phi(x)-\bar{u}_+| &\leq e^{f'(\bar{u}_+)(x-x_0)}\,|\phi(x_0)-\bar{u}_+| \\ &\quad +\int_{x_0}^{x} e^{f'(\bar{u}_+)(x-y)}\,|f(\phi(y))-f(\bar{u}_+)-f'(\bar{u}_+)(\phi(y)-\bar{u}_+)|\,dy \\ &= e^{f'(\bar{u}_+)(x-x_0)}\,|\phi(x_0)-\bar{u}_+| \\ &\quad +\int_{x_0}^{x} e^{f'(\bar{u}_+)(x-y)}\frac{1}{2}\,|f''((1-t)\phi(y)+t\bar{u}_+)|\,(\phi(y)-\bar{u}_+)^2 dy \\ &\leq e^{f'(\bar{u}_+)(x-x_0)}\,|\phi(x_0)-\bar{u}_+| \\ &\quad +\frac{1}{2}\delta C\int_{x_0}^{x} e^{f'(\bar{u}_+)(x-y)}\,|\phi(y)-\bar{u}_+|\,dy\end{aligned}$$

Multiply $e^{-f'(\bar{u}_+)(x-x_0)}$ on both sides

$$\begin{aligned}e^{-f'(\bar{u}_+)(x-x_0)}\,|\phi(x)-\bar{u}_+| &\leq |\phi(x_0)-\bar{u}_+| \\ &\quad +\frac{1}{2}\delta C\int_{x_0}^{x} e^{-f'(\bar{u}_+)(y-x_0)}\,|\phi(y)-\bar{u}_+|\,dy\end{aligned}$$

and apply Gronwall's Inequality to obtain the estimate

$$e^{-f'(\bar{u}_+)(x-x_0)}\,|\phi(x)-\bar{u}_+| \leq |\phi(x_0)-\bar{u}_+|\,e^{\frac{1}{2}\delta C(x-x_0)}$$

or equivalently

$$\begin{aligned}|\phi(x)-\bar{u}_+| &\leq |\phi(x_0)-\bar{u}_+|\,e^{(f'(\bar{u}_+)+\frac{1}{2}\delta C)(x-x_0)} \\ &\leq |\phi(x_0)-\bar{u}_+|\,e^{(f'(\bar{u}_+)+\delta C)(x-x_0)} \\ &= |\phi(x_0)-\bar{u}_+|\,e^{-\alpha(x-x_0)}\end{aligned}$$

$$|\phi(x)-\bar{u}_+| \leq |\phi(x_0)-\bar{u}_+|\,e^{-\alpha(x-x_0)} \tag{3.13}$$

Thus, if $|\phi(x_0)-\bar{u}_+|\leq\delta$ and $|\phi(x)-\bar{u}_+|<\delta$ for $x\in\mathfrak{J}$, then the required inequality (3.8) is satisfied for $x\in\mathfrak{J}$.
If $\mathfrak{J}$ is not the interval $[x_0,\infty)$, then $\xi<\infty$. Because $|\phi(x_0)-\bar{u}_+|\leq\delta$ and in view of inequality (3.13), we have that

$$|\phi(x)-\bar{u}_+| \leq \delta e^{-\alpha(x-x_0)} \tag{3.14}$$

for $x_0\leq x<\xi$. Therefore, by the extension theorem there is some number $\varepsilon>0$ such that the solution is defined on the interval $\mathfrak{J}':=[x_0,\xi+\varepsilon)$. Using the continuity of the function $x\mapsto|\phi(x)|$ on $\mathfrak{J}'$ and the inequality (3.14), it follows that

$$|\phi(\xi)-\bar{u}_+| \leq \delta e^{-\alpha(\xi-x_0)} < \delta \tag{3.15}$$

By using this inequality and again using the continuity of the function $x \mapsto |\phi(x)|$ on $\mathfrak{J}'$, there is a number $\kappa > 0$ such that $x \mapsto |\phi(x)|$ is defined on the interval $[x_0, \xi + \kappa)$, and, on this interval, $|\phi(\xi) - \bar{u}_+| < \delta$. This contradicts the maximality of ξ. Hence the inequality (3.8) is valid for all $x \geq x_0$. $\square$

Notice that we can rewrite (3.6) as

$$\bar{u}_x = f(\bar{u}) - f(\bar{u}_+) = f'(\xi)(\bar{u} - \bar{u}_+)$$

so we have the following estimate

$$|\bar{u}_x| \leq |f'|_{L^\infty} |\bar{u}(x_0) - \bar{u}_+| \, e^{-\alpha(x-x_0)}$$

We then can rephrase Lemma 3.1 as the following

Lemma 3.2. *Consider the initial value problem* (3.7) *with* $\bar{u}(x_0) = \bar{u}_0$. *If we assume that* $f'(\bar{u}_+) < 0$, *then there are positive constants* $\delta > 0$ *and* $\alpha > 0$ *that are independent of the choice of the initial time* x_0 *such that the solution* $x \mapsto \bar{u}(x)$ *of the initial value problem* (3.7) *satisfies*

$$|\bar{u}(x) - \bar{u}_+| \leq |\bar{u}(x_0) - \bar{u}_+| \, e^{-\alpha(x-x_0)} \tag{3.16}$$
$$|\bar{u}_x| \leq |f'|_{L^\infty} |\bar{u}(x_0) - \bar{u}_+| \, e^{-\alpha(x-x_0)} \tag{3.17}$$

for $x \geq x_0$ *whenever* $|\bar{u}(x_0) - \bar{u}_+| \leq \delta$.

Similarly, if we integrate (1.3) from $-\infty$ to some $x < 0$, we get

$$\bar{u}_x = f(\bar{u}) - f(\bar{u}_-) \tag{3.18}$$

Follow some similar arguments,

Lemma 3.3. *Consider the initial value problem* (3.18) *with* $\bar{u}(x_0) = \bar{u}_0$. *If we assume that* $f'(\bar{u}_-) > 0$, *then there are positive constants* $\delta > 0$ *and* $\alpha > 0$ *that are independent of the choice of the initial time* x_0 *such that the solution* $x \mapsto \bar{u}(x)$ *of the initial value problem* (3.18) *satisfies*

$$|\bar{u}(x) - \bar{u}_-| \leq |\bar{u}(x_0) - \bar{u}_-| \, e^{\alpha(x-x_0)} \tag{3.19}$$
$$|\bar{u}_x| \leq |f'|_{L^\infty} |\bar{u}(x_0) - \bar{u}_-| \, e^{\alpha(x-x_0)} \tag{3.20}$$

for $x \leq x_0$ *whenever* $|\bar{u}(x_0) - \bar{u}_-| \leq \delta$.

Combine Lemma 3.2 and 3.3 we know that if we assume the Lax condition

$$f'(\bar{u}_+) < 0 < f'(\bar{u}_-) \tag{3.21}$$

then

Proposition 3.4. *There are positive constants $\delta > 0$ and $\alpha > 0$ that are independent of the choice of the initial time x_0 such that the stationary wave solution $x \mapsto \bar{u}(x)$ satisfies asymptotic description.*

$$|\bar{u}(x) - \bar{u}_\pm| \leq |\bar{u}(x_0) - \bar{u}_\pm| \, e^{-\alpha|x-x_0|} \tag{3.22}$$

$$|\bar{u}_x| \leq |f'|_{L^\infty} \, |\bar{u}(x_0) - \bar{u}_\pm| \, e^{-\alpha|x-x_0|} \tag{3.23}$$

for $x \gtrless x_0$ whenever $|\bar{u}(x_0) - \bar{u}_\pm| \leq \delta$.

4 Gap Lemma

The "Gap Lemma" of [ZH] consists of the idea of relating the behavior near $x = \pm\infty$ of solutions of (2.2) to that of solutions of the asymptotic systems (2.5), in a manner that is analytic in λ. In this section, we state the general Gap Lemma for a general equation

$$W' = \mathbb{A}(x;\lambda)W \tag{4.1}$$

with the hypothesis:

$$|\mathbb{A} - \mathbb{A}_\pm| = \mathbf{O}(e^{-\alpha|x|}) \text{ as } x \to \pm\infty. \tag{4.2}$$

Proposition 4.1. *In (4.1), let $\mathbb{A}$ be $C^{0,\alpha}$ in x and analytic in λ, with $|\mathbb{A}(x;\lambda) - \mathbb{A}_-(\lambda)| = \mathbf{O}(e^{-\alpha|x|})$ as $x \to -\infty$ for $\alpha > 0$, and $0 < \bar{\alpha} < \alpha$. If $V^-(\lambda)$ is an eigenvector of $\mathbb{A}_-$ with eigenvalue $\mu(\lambda)$, both analytic in λ, then there exists a solution $W(x;\lambda)$ of (4.1) of form*

$$W(x;\lambda) = V(x;\lambda)e^{\mu(\lambda)x} \tag{4.3}$$

where V (hence W) is $C^{1,\alpha}$ in x and locally analytic in λ, and for each $j = 0, 1, 2, \ldots$ satisfies

$$\left(\frac{\partial}{\partial\lambda}\right)^j V(x;\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^j V^-(\lambda) + \mathbf{O}\left(e^{-\bar{\alpha}|x|}\left|\left(\frac{\partial}{\partial\lambda}\right)^j V^-(\lambda)\right|\right) \tag{4.4}$$

$$= \left(\frac{\partial}{\partial\lambda}\right)^j V^-(\lambda)(1 + \mathbf{O}(e^{-\bar{\alpha}|x|})) \tag{4.5}$$

for $x < 0$.
Moreover, if $\mathrm{Re}\mu(\lambda) > \mathrm{Re}\tilde{\mu}(\lambda) - \alpha$ for all eigenvalues $\tilde{\mu}(\lambda)$ of $\mathbb{A}_-(\lambda)$, then W is uniquely determined by (4.4), and (4.4) holds for $\bar{\alpha} = \alpha$.

5 Construction of the Resolvent Kernel

We now construct an explicit representation for the resolvent kernel, that is, the Green's function $G_\lambda(x,y)$ associated with the elliptic operator $(L-\lambda I)$, defined by

$$(L-\lambda I)G_\lambda(\cdot,y)=\delta_y I \tag{5.1}$$

where δ_y denotes the Dirac delta distribution centered at y. Let Λ be as defined in section 2. It is a standard fact that both the resolvent $(L-\lambda I)^{-1}$ and the Green's function $G_\lambda(x,y)$ are meromorphic in λ on Λ, with isolated poles of finite order.(See [He].) Using our explicit representation, we will show more, that $G_\lambda(x,y)$ in fact admits a meromorphic extension to a sector

$$\Omega_\theta := \{\lambda : \mathrm{Re}(\lambda) \geq -\theta_1 - \theta_2|\mathrm{Im}(\lambda)|\}; \theta_1, \theta_2 > 0,$$

On Λ, the following solutions

$$\phi^+(x) = W_1^+(x;\lambda) = V_1^+(x;\lambda)e^{\mu_1^+ x}, \text{ for } x>0 \text{ and,} \tag{5.2}$$
$$\phi^-(x) = W_2^-(x;\lambda) = V_2^-(x;\lambda)e^{\mu_2^- x}, \text{ for } x<0. \tag{5.3}$$

are all the decaying solutions of (2.2) at $x=\pm\infty$, respectively.
Then we can set our Green's function as

$$G_\lambda(x,y) = c_+\phi^+(x), \text{ for } x>y; \tag{5.4}$$
$$G_\lambda(x,y) = c_-\phi^-(x), \text{ for } x<y. \tag{5.5}$$

Use the jump condition

$$[G_\lambda]_{(y)} = c_+\phi^+(y) - c_-\phi^-(y) = 0, \tag{5.6}$$
$$[G_{\lambda,x}]_{(y)} = c_+\phi^+_x(y) - c_-\phi^-_x(y) = 1. \tag{5.7}$$

We solve for c^+ and c^- to have

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{1}{\phi^-(y)\phi^+_x(y)-\phi^+(y)\phi^-_x(y)}\begin{pmatrix} \phi^-(y) \\ \phi^+(y) \end{pmatrix}$$

thus we have

$$G_\lambda(x,y) = \frac{\phi^+(x)\phi^-(y)}{\phi^-(y)\phi^+_x(y)-\phi^+(y)\phi^-_x(y)}, \text{ for } x>y; \tag{5.8}$$
$$G_\lambda(x,y) = \frac{\phi^-(x)\phi^+(y)}{\phi^-(y)\phi^+_x(y)-\phi^+(y)\phi^-_x(y)}, \text{ for } x<y. \tag{5.9}$$

We now use another way to find the Green's function for the operator $L-\lambda I$. On Λ, the subspaces spanned by

$$
\begin{aligned}
\phi^+(x) &= W_1^+(x;\lambda) = V_1^+(x;\lambda)e^{\mu_1^+ x}, \text{ for } x>0 \text{ and,} && (5.10)\\
\phi^-(x) &= W_2^-(x;\lambda) = V_2^-(x;\lambda)e^{\mu_2^- x}, \text{ for } x<0. && (5.11)
\end{aligned}
$$

contain all solutions of (2.2) *decaying* at $x=\pm\infty$. We denote the complementary subspaces of *growing modes* by the subspace spanned by

$$
\begin{aligned}
\psi^+(x) &= W_2^+(x;\lambda) = V_2^+(x;\lambda)e^{\mu_2^+ x}, \text{ for } x>0 \text{ and,} && (5.12)\\
\psi^-(x) &= W_1^-(x;\lambda) = V_1^-(x;\lambda)e^{\mu_1^- x}, \text{ for } x<0. && (5.13)
\end{aligned}
$$

where $\mu_1^+ < 0 < \mu_2^+$ and $\mu_1^- < 0 < \mu_2^-$.
Eigenfunctions, decaying at both $\pm\infty$, occur precisely when the subspaces $\mathrm{Span}\{\phi^+\}$ and $\mathrm{Span}\{\phi^-\}$ intersect. This intersection can be detected by the vanishing of their mutual determinant, or equivalently of the *Evans function*,

$$
\begin{aligned}
D_L(\lambda) &:= \det(\phi^+,\phi^-)\,|_{x=0} && (5.14)\\
&= (\phi^+\wedge\phi^-)\,|_{x=0}\,. && (5.15)
\end{aligned}
$$

We now turn to the representation of the Green's function $G_\lambda(x,y)$.

Lemma 5.1. *Let $H_\lambda(x,y)$ denote the Green's function for the adjoint operator $(L-\lambda I)^*$. Then, $G_\lambda(y,x)=H_\lambda(x,y)^*$. In particular, for $x\neq y$, the matrix $z=G_\lambda(x,\cdot)$ satisfies*

$$z'' = -z'f'(\bar u)+\lambda z. \tag{5.16}$$

Proof. Notice that we have $(L-\lambda I)G_\lambda(x,y)=\delta_y(x)I$, and $(L-\lambda I)^*H_\lambda(x,y)=\delta_y(x)I$. So

$$
\begin{aligned}
G_\lambda(x_0,y_0) &= \langle \delta_{x_0}I, G_\lambda(x,y_0)\rangle\\
&= \langle (L-\lambda I)^*H_\lambda(x,x_0), G_\lambda(x,y_0)\rangle\\
&= \langle H_\lambda(x,x_0), (L-\lambda I)G_\lambda(x,y_0)\rangle\\
&= \langle H_\lambda(x,x_0), \delta_{y_0}(x)\rangle\\
&= H_\lambda(y_0,x_0)^*.
\end{aligned}
$$

□

The equation (5.16) is a adjoint equation of (2.1), written as a first order system by setting $Z=(z,z')$, it becomes

$$Z' = Z\tilde{\mathbb{A}}(x;\lambda), \tag{5.17}$$

where

$$\tilde{\mathbb{A}}(x;\lambda) := \begin{pmatrix} 0 & \lambda \\ 1 & -f'(\bar{u}) \end{pmatrix}. \tag{5.18}$$

The following lemma shows a duality relation between solutions of (2.2) and solutions of (5.17).

Lemma 5.2. *Z is a solution of* (5.17) *if and only if $Z\mathcal{S}W \equiv$ constant for any solution W of* (2.2)*, where $\mathcal{S} = \begin{pmatrix} -f'(\bar{u}) & 1 \\ -1 & 0 \end{pmatrix}$.*

Proof.

$$\begin{aligned}
(Z\mathcal{S}W)' &= (-zf'(\bar{u})w - z'w + zw')' \\
&= -z'f'(\bar{u})w - z(f'(\bar{u})w)' - z''w - z'w' + z'w' + zw'' \\
&= -z'f'(\bar{u})w - z(f'(\bar{u})w)' - z''w + zw'' \\
&= z[w'' - (f'(\bar{u})w)'] - [z'' + z'f'(\bar{u})]w \\
&= z[w'' - (f'(\bar{u})w)' - \lambda w] - [z'' + z'f'(\bar{u}) - \lambda z]w \\
&= [z'' + z'f'(\bar{u}) - \lambda z]w
\end{aligned}$$

since w is a solution of (2.1), thus z is a solution of (5.16) if and only if $(Z\mathcal{S}W)' \equiv 0$. □

Using Lemma 5.2, we can immediately define dual bases $\tilde{W}_1^{\pm}$ and $\tilde{W}_2^{\pm}$ of solutions to (5.17) by the relation

$$\tilde{W}_j^{\pm}\mathcal{S}W_k^{\pm} = \delta_{jk}, \tag{5.19}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

denotes the Kronecker delta function. Note that we have defined the $\tilde{W}_j^{\pm}$ as row vectors. And $\tilde{W}_j^{\pm}$ has the following form

$$\tilde{W}_j^{\pm}(x;\lambda) = \tilde{V}_j^{\pm}(x;\lambda)e^{\tilde{\mu}_j^{\pm}x} \tag{5.20}$$

According to (5.19), we have

$$\begin{cases} \tilde{\mu}_j^{\pm} = -\mu_j^{\pm} \\ \tilde{V}_j^{\pm}\mathcal{S}V_k^{\pm} = \delta_{jk} \end{cases}$$

for all λ. In accordance with (5.10)-(5.12), we define the dual subspace spanned by

$$\begin{aligned} \tilde{\phi}^+(x) &= \tilde{W}_1^+(x;\lambda) = \tilde{V}_1^+(x;\lambda)e^{\tilde{\mu}_1^+ x} = \tilde{V}_1^+(x;\lambda)e^{-\mu_1^+ x}, \text{ for } x > 0 \quad (5.21) \\ \tilde{\phi}^-(x) &= \tilde{W}_2^-(x;\lambda) = \tilde{V}_2^-(x;\lambda)e^{\tilde{\mu}_2^- x} = \tilde{V}_2^-(x;\lambda)e^{-\mu_2^- x}, \text{ for } x < 0 \quad (5.22) \end{aligned}$$

as the *growing subspace* and the dual subspace spanned by

$$\begin{aligned} \tilde{\psi}^+(x) &= \tilde{W}_2^+(x;\lambda) = \tilde{V}_2^+(x;\lambda)e^{\tilde{\mu}_2^+ x} = \tilde{V}_2^+(x;\lambda)e^{-\mu_2^+ x}, \text{ for } x > 0 \quad (5.23) \\ \tilde{\psi}^-(x) &= \tilde{W}_1^-(x;\lambda) = \tilde{V}_1^-(x;\lambda)e^{\tilde{\mu}_1^- x} = \tilde{V}_1^-(x;\lambda)e^{-\mu_1^- x}, \text{ for } x < 0 \quad (5.24) \end{aligned}$$

as the *decaying subspace*. We may define dual exponentially decaying and growing solutions $\tilde{\phi}^\pm$ and $\tilde{\psi}^\pm$ via

$$\begin{aligned} &\tilde{\phi}^\pm \mathcal{S}\phi^\pm = 1; \quad \tilde{\phi}^\pm \mathcal{S}\psi^\pm = 0, \\ &\tilde{\psi}^\pm \mathcal{S}\phi^\pm = 0; \quad \tilde{\psi}^\pm \mathcal{S}\psi^\pm = 1. \end{aligned} \tag{5.25}$$

or written as matrix form:

$$\begin{pmatrix} \tilde{\phi}^\pm \\ \tilde{\psi}^\pm \end{pmatrix} \mathcal{S} \left(\phi^\pm, \psi^\pm\right) = I.$$

From $(L-\lambda I)G_\lambda(x,y) = \delta_y(x)I$, $(L-\lambda I)^* H_\lambda(x,y) = \delta_y(x)I$ and Lemma 5.1 we know that $\begin{pmatrix} G_\lambda(x,y) \\ G_{\lambda,x}(x,y) \end{pmatrix}$ viewed as a function of x satisfies (2.2)(differentiating with respect to x), while $(G_\lambda(x,y), G_{\lambda,y}(x,y))$ viewed as function of y satisfies (5.17)(differentiating with respect to y). Further, note that both $G_\lambda(x,\cdot)$ and $G_\lambda(\cdot,y)$ decay at $\pm\infty$ for λ on the resolvent set, since $|(L-\lambda I)^{-1}| < \infty$ and $|(L-\lambda I)^{*-1}| < \infty$ imply $\|G_\lambda(\cdot,y)\|_{L^1(x)} < \infty$ and $\|G_\lambda(x,\cdot)\|_{L^1(y)} < \infty$ respectively. Combining, we have the representation

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = \begin{cases} \phi^+(x;\lambda)m^+(\lambda)\tilde{\psi}^-(y;\lambda) & \text{for } x > y; \\ -\phi^-(x;\lambda)m^-(\lambda)\tilde{\psi}^+(y;\lambda) & \text{for } x < y, \end{cases} \tag{5.26}$$

where numbers $m^\pm(\lambda)$ are to be determined.

Lemma 5.3.

$$\begin{bmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{bmatrix}_{(y)} = \begin{pmatrix} 0 & -1 \\ 1 & -f'(\bar{u}) \end{pmatrix} = \mathcal{S}^{-1},$$

where $[f(x)]_{(y)}$ *denotes the jump in* $f(x)$ *at* $x = y$*, and* $\mathcal{S}$ *is as in* Lemma 5.2.

Proof. Expanding $\delta_y(x) = (L - \lambda I)G_\lambda = G_{\lambda,xx} - (f'(\bar{u})G_\lambda)_x - \lambda G_\lambda$, and comparing orders of singularity, we find that

$$(f'(\bar{u})G_\lambda)_x + \lambda G_\lambda = 0 \quad \text{and} \quad G_{\lambda,xx} = \delta_y(x),$$

giving, respectively,

$$[G_\lambda]_{(y)} = 0 \quad \text{and} \quad [G_{\lambda,x}]_{(y)} = 1,$$

Note further that we can expand $[G_\lambda]_{(y)}$ as

$$[G_\lambda]_{(y)} = [G_\lambda(\cdot, y)]_{(y)} = G_\lambda^{x>y}(y,y) - G_\lambda^{x<y}(y,y), \tag{5.27}$$

where $G_\lambda^{x>y}$ and $G_\lambda^{x<y}$ are the smooth functions denoting the value of G_λ on the regions $x > y$ and $x < y$, respectively, i.e.

$$G_\lambda(x,y) = \begin{cases} G_\lambda^{x>y}(x,y) & \text{for } x > y; \\ G_\lambda^{x<y}(x,y) & \text{for } x < y, \end{cases}$$

Differentiating (5.27) in y, we obtain

$$\begin{aligned} 0 = \frac{d}{dy}[G_\lambda]_{(y)} &= G_{\lambda,x}^{x>y}(y,y) + G_{\lambda,y}^{x>y}(y,y) - G_{\lambda,x}^{x<y}(y,y) - G_{\lambda,y}^{x<y}(y,y) \\ &= [G_{\lambda,x}]_{(y)} + [G_{\lambda,y}]_{(y)} \end{aligned}$$

hence

$$[G_{\lambda,y}]_{(y)} = -1.$$

Differentiating $0 = [G_{\lambda,x}]_{(y)} + [G_{\lambda,y}]_{(y)}$ a second time, we then find

$$\begin{aligned} 0 &= [G_{\lambda,xx}]_{(y)} + [G_{\lambda,xy}]_{(y)} + [G_{\lambda,yx}]_{(y)} + [G_{\lambda,yy}]_{(y)} \\ &= [G_{\lambda,xx}]_{(y)} + [G_{\lambda,yy}]_{(y)} + 2[G_{\lambda,xy}]_{(y)} \end{aligned}$$

Solve for $[G_{\lambda,xy}]_{(y)}$ to find that

$$[G_{\lambda,xy}]_{(y)} = -\frac{1}{2}\left([G_{\lambda,xx}]_{(y)} + [G_{\lambda,yy}]_{(y)}\right). \tag{5.28}$$

Finally, we can determine $[G_{\lambda,xx}]_{(y)}$ and $[G_{\lambda,yy}]_{(y)}$ by solving the ODE (2.1) and (5.16) to express

$$\begin{aligned} G_{\lambda,xx} &= f'(\bar{u})G_{\lambda,x} + (f'(\bar{u}))_x G_\lambda; \\ G_{\lambda,yy} &= -f'(\bar{u})G_{\lambda,y} + \lambda G_\lambda. \end{aligned}$$

and then

$$\begin{aligned}
[G_{\lambda,xx}]_{(y)} &= f'(\bar{u}(y))\,[G_{\lambda,x}]_{(y)} + f''(\bar{u}(y))\bar{u}'(y)\,[G_\lambda]_{(y)} = f'(\bar{u}(y));\\
[G_{\lambda,yy}]_{(y)} &= -f'(\bar{u}(y))\,[G_{\lambda,y}]_{(y)} + \lambda\,[G_\lambda]_{(y)} = f'(\bar{u}(y)).
\end{aligned}$$

so finally (5.28) gives

$$[G_{\lambda,xy}]_{(y)} = -f'(\bar{u}(y)).$$

Then

$$\begin{bmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{bmatrix}_{(y)} = \begin{pmatrix} 0 & -1 \\ 1 & -f'(\bar{u}) \end{pmatrix} = \mathcal{S}^{-1},$$

as claimed. □

Combining Lemma 5.3 with (5.26), we have

$$\left(\phi^+(y), \phi^-(y)\right) \begin{pmatrix} m^+(\lambda) & 0 \\ 0 & m^-(\lambda) \end{pmatrix} \begin{pmatrix} \tilde{\psi}^-(y) \\ \tilde{\psi}^+(y) \end{pmatrix} = \mathcal{S}^{-1}, \quad \text{or}$$

$$\begin{aligned}
\begin{pmatrix} m^+(\lambda) & 0 \\ 0 & m^-(\lambda) \end{pmatrix} &= \left(\phi^+(y), \phi^-(y)\right)^{-1} \mathcal{S}^{-1} \begin{pmatrix} \tilde{\psi}^-(y) \\ \tilde{\psi}^+(y) \end{pmatrix}^{-1} \\
&= \left(\begin{pmatrix} \tilde{\psi}^-(y) \\ \tilde{\psi}^+(y) \end{pmatrix} \mathcal{S} \left(\phi^+(y), \phi^-(y)\right) \right)^{-1} \\
&= \begin{pmatrix} \tilde{\psi}^-\mathcal{S}\phi^+ & \tilde{\psi}^-\mathcal{S}\phi^- \\ \tilde{\psi}^+\mathcal{S}\phi^+ & \tilde{\psi}^+\mathcal{S}\phi^- \end{pmatrix}^{-1} (y) \\
&= \begin{pmatrix} \tilde{\psi}^-\mathcal{S}\phi^+ & 0 \\ 0 & \tilde{\psi}^+\mathcal{S}\phi^- \end{pmatrix}^{-1} (y) \\
&= \begin{pmatrix} \frac{1}{\tilde{\psi}^-\mathcal{S}\phi^+} & 0 \\ 0 & \frac{1}{\tilde{\psi}^+\mathcal{S}\phi^-} \end{pmatrix}
\end{aligned}$$

so

$$m^+(\lambda) = \frac{1}{\tilde{\psi}^-\mathcal{S}\phi^+} = \frac{1}{\tilde{V}_1^-(x;\lambda)\mathcal{S}V_1^+(x;\lambda)e^{(\mu_1^+-\mu_1^-)x}} \tag{5.29}$$

$$m^-(\lambda) = \frac{1}{\tilde{\psi}^+\mathcal{S}\phi^-} = \frac{1}{\tilde{V}_2^+(x;\lambda)\mathcal{S}V_2^-(x;\lambda)e^{(\mu_2^--\mu_2^+)x}} \tag{5.30}$$

We now introduce the notation,

$$\begin{aligned}
\Phi : &= \left(\phi^+, \phi^-\right) \text{ then } \Phi \text{ is a } 2\times 2 \text{ matrix}; && (5.31)\\
\Psi : &= \left(\psi^-, \psi^+\right); && (5.32)\\
\tilde{\Psi} : &= \begin{pmatrix} \tilde{\psi}^- \\ \tilde{\psi}^+ \end{pmatrix}; && (5.33)\\
\tilde{\Phi} : &= \begin{pmatrix} \tilde{\phi}^+ \\ \tilde{\phi}^- \end{pmatrix}. && (5.34)
\end{aligned}$$

Proposition 5.4. *The resolvent kernel may be expressed as*

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = \begin{cases} \phi^+(x;\lambda) m^+(\lambda) \tilde{\psi}^-(y;\lambda) & \text{for} \quad x > y; \\ -\phi^-(x;\lambda) m^-(\lambda) \tilde{\psi}^+(y;\lambda) & \text{for} \quad x < y, \end{cases} \tag{5.35}$$

where

$$M(\lambda) := \operatorname{diag}(m^+(\lambda), m^-(\lambda)) = \Phi^{-1}(z;\lambda)\mathcal{S}^{-1}(z)\tilde{\Psi}^{-1}(z;\lambda) \tag{5.36}$$

From Proposition 5.4, we obtain the following scattering decomposition,

Corollary 5.5. *On $\Lambda \cap \rho(L)$,*

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = m^+(\lambda)\phi^+(x;\lambda)\tilde{\psi}^-(y;\lambda) \tag{5.37}$$

for $y \le 0 \le x$,

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = d^+(\lambda)\phi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) + \psi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) \tag{5.38}$$

for $y \le x \le 0$, where

$$m^+ = (1,0)(\phi^+, \phi^-)^{-1}\psi^- \tag{5.39}$$

and

$$d^+ = -(0,1)(\phi^+, \phi^-)^{-1}\psi^- \tag{5.40}$$

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = -m^-(\lambda)\phi^-(x;\lambda)\tilde{\psi}^+(y;\lambda) \tag{5.41}$$

for $x \le 0 \le y$,

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} = d^-(\lambda)\phi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) - \phi^-(x;\lambda)\tilde{\phi}^-(y;\lambda) \tag{5.42}$$

for $x \le y \le 0$, where

$$m^- = \tilde{\phi}^- \begin{pmatrix} \tilde{\psi}^- \\ \tilde{\psi}^+ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.43}$$

and

$$d^-(\lambda) = \tilde{\phi}^- \begin{pmatrix} \tilde{\psi}^- \\ \tilde{\psi}^+ \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.44}$$

Proof. We may express m^+ using the duality relation (5.25) as

$$\begin{aligned} m^+ &= (1,0)(\phi^+,\phi^-)^{-1}\mathcal{S}^{-1} \begin{pmatrix} \tilde{\psi}^- \\ \tilde{\phi}^- \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1,0)(\phi^+,\phi^-)^{-1} \left(\begin{pmatrix} \tilde{\psi}^- \\ \tilde{\phi}^- \end{pmatrix} \mathcal{S} \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1,0)(\phi^+,\phi^-)^{-1}(\psi^-,\phi^-) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (1,0)(\phi^+,\phi^-)^{-1}\psi^- \end{aligned}$$

Next, expressing $\phi^+(x;\lambda)$ as a linear combination of basis elements at $-\infty$,

$$\begin{aligned} \phi^+(x;\lambda) &= a^+(\lambda)\phi^-(x;\lambda) + b^+(\lambda)\psi^-(x;\lambda) \\ &= (\phi^-,\psi^-) \begin{pmatrix} a^+ \\ b^+ \end{pmatrix} \end{aligned}$$

so we get

$$\begin{pmatrix} a^+ \\ b^+ \end{pmatrix} = (\phi^-,\psi^-)^{-1}\phi^+(x;\lambda) = \begin{pmatrix} \tilde{\phi}^- \\ \tilde{\psi}^- \end{pmatrix} \mathcal{S}\phi^+$$

then we can represent $\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix}$ as

$$\begin{aligned} \begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} &= m^+(\lambda)\phi^+(x;\lambda)\tilde{\psi}^-(y;\lambda) \\ &= m^+(\lambda)[a^+(\lambda)\phi^-(x;\lambda) + b^+(\lambda)\psi^-(x;\lambda)]\tilde{\psi}^-(y;\lambda) \\ &= a^+(\lambda)m^+(\lambda)\phi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) + b^+(\lambda)m^+(\lambda)\psi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) \\ &= d^+(\lambda)\phi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) + e^+(\lambda)\psi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) \end{aligned}$$

where $d^+ = a^+m^+$ and $e^+ = b^+m^+$, and can be computed as follows,

$$
\begin{aligned}
\begin{pmatrix} d^+ \\ e^+ \end{pmatrix} &= \begin{pmatrix} a^+ \\ b^+ \end{pmatrix} m^+ \\
&= \begin{pmatrix} a^+ \\ b^+ \end{pmatrix} (1,0)(\phi^+,\phi^-)^{-1}\psi^- \\
&= \begin{pmatrix} \tilde{\phi}^- \\ \tilde{\psi}^- \end{pmatrix} \mathcal{S}\phi^+(1,0)(\phi^+,\phi^-)^{-1}\psi^- \\
&= \begin{pmatrix} \tilde{\phi}^- \\ \tilde{\psi}^- \end{pmatrix} \mathcal{S}(\phi^+,0)(\phi^+,\phi^-)^{-1}\psi^- \\
&= (\phi^-,\psi^-)^{-1}(\phi^+,0)(\phi^+,\phi^-)^{-1}\psi^- \\
&= (\phi^-,\psi^-)^{-1}\left(I_2 - (0,\phi^-)(\phi^+,\phi^-)^{-1}\right)\psi^- \\
&= (\phi^-,\psi^-)^{-1}\psi^- - (\phi^-,\psi^-)^{-1}(0,\phi^-)(\phi^+,\phi^-)^{-1}\psi^- \\
&= (\phi^-,\psi^-)^{-1}(\phi^-,\psi^-)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - (\phi^-,\psi^-)^{-1}(\phi^-,\psi^-)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(\phi^+,\phi^-)^{-1}\psi^- \\
&= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(\phi^+,\phi^-)^{-1}\psi^-
\end{aligned}
$$

yielding (5.38) and (5.40). □

6 Low-frequency expanions

Lemma 6.1. *For $\lambda \in \Lambda$, the matrix $\mathbb{A}_\pm(\lambda)$ in (2.5) has eigenvalues $\mu_1^\pm(\lambda) < 0 < \mu_2^\pm(\lambda)$ (with ordering referring to real parts) such that the eigenspaces $S^\pm(\lambda)$ and $U^\pm(\lambda)$ associated to $\mu_1^\pm(\lambda)$ and $\mu_2^\pm(\lambda)$ respectively depend analytically on λ. Furthermore, for each $j = 1,2$, there is an analytic extension of $\mu_j^\pm(\lambda)$ to some neighborhood N of $\lambda = 0$. For $\lambda \in N$ there also exists an analytic choice of an individual eigenvector $V_j^\pm = \begin{pmatrix} v_j^\pm \\ \mu_j^\pm v_j^\pm \end{pmatrix}$ corresponding to each eigenvalue $\mu_j^\pm(\lambda)$, and they satisfy the following asymptotic descriptions:*

$$
\mu_1^\pm(\lambda) = \begin{cases} -\frac{\lambda}{f'(\bar{u}_\pm)} + \frac{\lambda^2}{(f'(\bar{u}_\pm))^3} + \mathbf{O}(\lambda^3), & \text{if} \quad f'(\bar{u}_\pm) > 0; \\ \\ f'(\bar{u}_\pm) + \mathbf{O}(\lambda), & \text{if} \quad f'(\bar{u}_\pm) < 0, \end{cases} \tag{6.1}
$$

$$
\mu_2^\pm(\lambda) = \begin{cases} -\frac{\lambda}{f'(\bar{u}_\pm)} + \frac{\lambda^2}{(f'(\bar{u}_\pm))^3} + \mathbf{O}(\lambda^3), & \text{if} \quad f'(\bar{u}_\pm) < 0; \\ \\ f'(\bar{u}_\pm) + \mathbf{O}(\lambda), & \text{if} \quad f'(\bar{u}_\pm) > 0, \end{cases} \tag{6.2}
$$

$$V_1^{\pm}(\lambda) = \begin{cases} \begin{pmatrix} 1+\mathbf{O}(\lambda) \\ -\frac{\lambda}{f'(\bar{u}_\pm)} + \mathbf{O}(\lambda^2) \end{pmatrix}, & \text{if} \quad f'(\bar{u}_\pm) > 0; \\ \\ \begin{pmatrix} 1 \\ f'(\bar{u}_\pm) \end{pmatrix} + \mathbf{O}(\lambda), & \text{if} \quad f'(\bar{u}_\pm) < 0, \end{cases} \tag{6.3}$$

$$V_2^{\pm}(\lambda) = \begin{cases} \begin{pmatrix} 1+\mathbf{O}(\lambda) \\ -\frac{\lambda}{f'(\bar{u}_\pm)} + \mathbf{O}(\lambda^2) \end{pmatrix}, & \text{if} \quad f'(\bar{u}_\pm) < 0; \\ \\ \begin{pmatrix} 1 \\ f'(\bar{u}_\pm) \end{pmatrix} + \mathbf{O}(\lambda), & \text{if} \quad f'(\bar{u}_\pm) > 0, \end{cases} \tag{6.4}$$

The spectral projection operators $P_{S^{\pm}(\lambda)}$ and $P_{U^{\pm}(\lambda)}$ associated to the subspaces $S^{\pm}(\lambda)$ and $U^{\pm}(\lambda)$ have analytic extensions to the neighborhood $\Omega = \{\lambda : \mathrm{Re}\lambda > 0\} \cup N$.

Since we have assumed the Lax condition (3.21),

$$f'(\bar{u}_+) < 0 < f'(\bar{u}_-)$$

so the results in above lemma simplify to read

$$\begin{aligned} \mu_1^+(\lambda) &= f'(\bar{u}_+) + \mathbf{O}(\lambda), && (6.5) \\ \mu_2^+(\lambda) &= -\frac{\lambda}{f'(\bar{u}_+)} + \frac{\lambda^2}{(f'(\bar{u}_+))^3} + \mathbf{O}(\lambda^3), && (6.6) \\ \mu_1^-(\lambda) &= -\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3), && (6.7) \\ \mu_2^-(\lambda) &= f'(\bar{u}_-) + \mathbf{O}(\lambda), && (6.8) \\ V_1^+(\lambda) &= \begin{pmatrix} 1 \\ f'(\bar{u}_+) \end{pmatrix} + \mathbf{O}(\lambda), && (6.9) \\ V_2^+(\lambda) &= \begin{pmatrix} 1+\mathbf{O}(\lambda) \\ -\frac{\lambda}{f'(\bar{u}_+)} + \mathbf{O}(\lambda^2) \end{pmatrix}, && (6.10) \\ V_1^-(\lambda) &= \begin{pmatrix} 1+\mathbf{O}(\lambda) \\ -\frac{\lambda}{f'(\bar{u}_-)} + \mathbf{O}(\lambda^2) \end{pmatrix}, && (6.11) \\ V_2^-(\lambda) &= \begin{pmatrix} 1 \\ f'(\bar{u}_-) \end{pmatrix} + \mathbf{O}(\lambda), && (6.12) \end{aligned}$$

Lemma 6.2. *For $\lambda \in \Omega \cap \{\lambda : |\lambda| < \delta\}$ and δ sufficiently small, there exist solutions $W_j^{\pm}(x;\lambda)$ of (2.2), $(j = 1, 2)$, C^1 in x and analytic in λ, satisfying*

$$W_j^{\pm}(x;\lambda) = V_j^{\pm}(x;\lambda) e^{\mu_j^{\pm}(\lambda)x} \tag{6.13}$$

$$\left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(x;\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|x|}\left|\left(\frac{\partial}{\partial\lambda}\right)^k V_j^{\pm}(\lambda)\right|\right) \quad (6.14)$$

for any $k \geq 0$ and $0 < \tilde{\alpha} < \alpha$, where α is the rate of decay given in Proposition 3.4*, $\mu_j^{\pm}(\lambda)$ and $V_j^{\pm}(\lambda)$ are as above, and $\mathbf{O}(\cdot)$ depends only on $k, \tilde{\alpha}$.*

Proof. This is a direct consequence of the Gap Lemma(Proposition 4.1). □

Now we classify our forward and dual modes in a greater detail:
The *fast growing modes of the dual problem* (5.17) are

$$\begin{aligned} \tilde{\phi}^+(x) &= \tilde{W}_1^+(x;\lambda) = \tilde{V}_1^+(x;\lambda)e^{-\mu_1^+(\lambda)x}, & (6.15)\\ \tilde{\phi}^-(x) &= \tilde{W}_2^-(x;\lambda) = \tilde{V}_2^-(x;\lambda)e^{-\mu_2^-(\lambda)x}, & (6.16)\end{aligned}$$

the *slow decaying modes of the dual problem* (5.17) are

$$\tilde{\psi}^+(x) = \tilde{W}_2^+(x;\lambda) = \tilde{V}_2^+(x;\lambda)e^{-\mu_2^+(\lambda)x}, \quad (6.17)$$

$$\tilde{\psi}^-(x) = \tilde{W}_1^-(x;\lambda) = \tilde{V}_1^-(x;\lambda)e^{-\mu_1^-(\lambda)x}, \quad (6.18)$$

the *fast decaying modes of the forward problem* (2.2) are

$$\begin{aligned} \phi^+(x) &= W_1^+(x;\lambda) = V_1^+(x;\lambda)e^{\mu_1^+(\lambda)x}, & (6.19)\\ \phi^-(x) &= W_2^-(x;\lambda) = V_2^-(x;\lambda)e^{\mu_2^-(\lambda)x}, & (6.20)\end{aligned}$$

the *slow growing modes of the forward problem* (2.2) are

$$\begin{aligned} \psi^+(x) &= W_2^+(x;\lambda) = V_2^+(x;\lambda)e^{\mu_2^+(\lambda)x}, & (6.21)\\ \psi^-(x) &= W_1^-(x;\lambda) = V_1^-(x;\lambda)e^{\mu_1^-(\lambda)x}. & (6.22)\end{aligned}$$

Specifically, due to the special, conservative structure of the underlying evolution equations, the adjoint eigenvalue equation (dual problem (5.17)) at $\lambda = 0$ can be written as

$$\tilde{W}' = \tilde{W}\begin{pmatrix} 0 & 0 \\ 1 & -f'(\bar{u}) \end{pmatrix} \quad (6.23)$$

so it admits a 1−dimensional subspace of constant solutions

$$\tilde{W} \equiv (c, 0)$$

where c is a constant. Thus, at $\lambda = 0$, we may choose, by appropriate change of coordinates if necessary, to have slow decaying dual modes $\tilde{\psi}^{\pm}(x)$((6.17)-(6.18)) identically constant. Because when we let $\lambda = 0$ in (6.17)-(6.18), we have

$$\begin{aligned} \tilde{\psi}^{+}(x) &= \tilde{W}_2^{+}(x;0) = \tilde{V}_2^{+}(x;0)e^{-\mu_2^{+}(0)x} = \tilde{V}_2^{+}(x;0), \\ \tilde{\psi}^{-}(x) &= \tilde{W}_1^{-}(x;0) = \tilde{V}_1^{-}(x;0)e^{-\mu_1^{-}(0)x} = \tilde{V}_1^{-}(x;0), \end{aligned}$$

and we can choose $\tilde{V}_2^{+}(x;0) \equiv$ constant and $\tilde{V}_1^{-}(x;0) \equiv$ constant according to the above observation.

Lemma 6.3. *With the above choice of bases at $\lambda = 0$, and for $\lambda \in \Omega \cap \{\lambda : |\lambda| < \delta\}$ and δ sufficiently small, slow decaying dual modes $\tilde{W}_2^{+}(y;\lambda)$ and $\tilde{W}_1^{-}(y;\lambda)$ satisfy*

$$\tilde{W}_j^{\pm}(y;\lambda) = e^{-\mu_j^{\pm}(\lambda)y}\tilde{V}_j^{\pm}(0) + \lambda\tilde{\Theta}_j^{\pm}(y;\lambda), \tag{6.24}$$

where

$$\left|\tilde{\Theta}_j^{\pm}\right| \le C\left|e^{-\mu_j^{\pm}(\lambda)y}\right|, \tag{6.25}$$

$$\left|\left(\frac{\partial}{\partial y}\right)\tilde{\Theta}_j^{\pm}\right| \le C\left|e^{-\mu_j^{\pm}(\lambda)y}\right|\left(|\lambda| + e^{-\alpha|y|}\right), \tag{6.26}$$

$\alpha > 0$ is the rate of decay given in Proposition 3.4*, as $y \to \pm\infty$, and $\tilde{V}_j^{\pm}(0) \equiv$ constant.*
Similarly, fast decaying forward modes $W_1^{+}(x;\lambda)$ and $W_2^{-}(x;\lambda)$ satisfy

$$W_j^{\pm}(x;\lambda) = W_j^{\pm}(x;0) + \lambda\Theta_j^{\pm}(x;\lambda), \tag{6.27}$$

where

$$\begin{aligned} \left|\Theta_j^{\pm}\right| &\le Ce^{-\alpha|x|}, &(6.28)\\ \left|\left(\frac{\partial}{\partial y}\right)\Theta_j^{\pm}\right| &\le Ce^{-\alpha|x|}, &(6.29) \end{aligned}$$

as $x \to \pm\infty$.

Proof. First, let us consider the augmented variables

$$\begin{aligned} \tilde{\mathbb{W}}_j^{\pm}(y;\lambda) : &= \left(\tilde{W}_j^{\pm}, \tilde{W}_j^{\pm\prime}\right)(y;\lambda) \\ &= e^{-\mu_j^{\pm}(\lambda)y}\tilde{\mathbb{V}}_j^{\pm}(y;\lambda) \\ &= e^{-\mu_j^{\pm}(\lambda)y}\left(\tilde{V}_j^{\pm}, -\mu_j^{\pm}\tilde{V}_j^{\pm} + \tilde{V}_j^{\pm\prime}\right)(y;\lambda) \end{aligned}$$

and

$$
\begin{aligned}
\mathbb{W}_j^{\pm}(x;\lambda) : &= \begin{pmatrix} W_j^{\pm} \\ W_j^{\pm\prime} \end{pmatrix}(x;\lambda) \\
&= e^{\mu_j^{\pm}(\lambda)x}\mathbb{V}_j^{\pm}(x;\lambda) \\
&= e^{\mu_j^{\pm}(\lambda)x}\begin{pmatrix} V_j^{\pm} \\ \mu_j^{\pm}V_j^{\pm} + V_j^{\pm\prime} \end{pmatrix}(x;\lambda)
\end{aligned}
$$

Note that since $W_j^{\pm}(x;\lambda)$ satisfies $W' = \mathbb{A}(x;\lambda)W$, so $W'' = \mathbb{A}'W + \mathbb{A}W'$ and so

$$
\mathbb{W}(x;\lambda) = \begin{pmatrix} W \\ W' \end{pmatrix}(x;\lambda)
$$

satisfies

$$
\mathbb{W}' = \begin{pmatrix} \mathbb{A} & 0 \\ \mathbb{A}' & \mathbb{A} \end{pmatrix}(x;\lambda)\mathbb{W}
$$

Let $x \to \pm\infty$ in the coefficient matrix above, we get the limiting equation

$$
\mathbb{W}' = \begin{pmatrix} \mathbb{A}_{\pm} & 0 \\ 0 & \mathbb{A}_{\pm} \end{pmatrix}(\lambda)\mathbb{W}
$$

If $\bar{W}_j^{\pm}(x;\lambda) = V_j^{\pm}(\lambda)e^{\mu_j^{\pm}(\lambda)x}$ is a solution of $W' = \mathbb{A}_{\pm}W$, then

$$
\begin{aligned}
\bar{\mathbb{W}}_j^{\pm}(x;\lambda) &= \begin{pmatrix} \bar{W}_j^{\pm}(x;\lambda) \\ \bar{W}_j^{\pm\prime}(x;\lambda) \end{pmatrix} \\
&= e^{\mu_j^{\pm}(\lambda)x}\begin{pmatrix} V_j^{\pm}(\lambda) \\ \mu_j^{\pm}(\lambda)V_j^{\pm}(\lambda) \end{pmatrix}
\end{aligned}
$$

is a solution of

$$
\mathbb{W}' = \begin{pmatrix} \mathbb{A}_{\pm} & 0 \\ 0 & \mathbb{A}_{\pm} \end{pmatrix}(\lambda)\mathbb{W}
$$

Now we apply the Gap Lemma to obtain bounds

$$
\tilde{\mathbb{W}}_j^{\pm}(y;\lambda) = \tilde{\mathbb{V}}_j^{\pm}(y;\lambda)e^{-\mu_j^{\pm}(\lambda)y}, \tag{6.30}
$$

$$
\left(\frac{\partial}{\partial\lambda}\right)^k \tilde{\mathbb{V}}_j^{\pm}(y;\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^k \tilde{\mathbb{V}}_j^{\pm}(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|y|}\left|\tilde{\mathbb{V}}_j^{\pm}(\lambda)\right|\right), y \gtrless 0, \tag{6.31}
$$

and

$$
\mathbb{W}_j^{\pm}(x;\lambda) = \mathbb{V}_j^{\pm}(x;\lambda)e^{\mu_j^{\pm}(\lambda)x}, \tag{6.32}
$$

$$\left(\frac{\partial}{\partial\lambda}\right)^k \mathbb{V}_j^\pm(x;\lambda) = \left(\frac{\partial}{\partial\lambda}\right)^k \mathbb{V}_j^\pm(\lambda) + \mathbf{O}\left(e^{-\tilde\alpha|x|}\left|\mathbb{V}_j^\pm(\lambda)\right|\right), x \gtrless 0, \qquad (6.33)$$

$0 < \tilde\alpha < \alpha$, analogous to Lemma 6.2, valid for $\lambda \in \Omega \cap \{\lambda : |\lambda| < \delta\}$, where

$$\tilde{\mathbb{V}}_j^\pm(\lambda) = \left(\tilde V_j^\pm(\lambda), -\mu_j^\pm(\lambda)\tilde V_j^\pm(\lambda)\right)$$

and

$$\mathbb{V}_j^\pm(\lambda) = \begin{pmatrix} V_j^\pm(\lambda) \\ \mu_j^\pm(\lambda)V_j^\pm(\lambda)\end{pmatrix}$$

By Taylor's Theorem with differential remainder, applied to $\tilde{\mathbb{V}}_j^\pm(y;\lambda)$ with respect to λ, we have:

$$\tilde{\mathbb{W}}_j^\pm(y;\lambda) = e^{-\mu_j^\pm(\lambda)y}\left(\tilde{\mathbb{V}}_j^\pm(y;0) + \lambda\left(\frac{\partial}{\partial\lambda}\right)\tilde{\mathbb{V}}_j^\pm(y;0) + \frac{1}{2}\lambda^2\left(\frac{\partial}{\partial\lambda}\right)^2\tilde{\mathbb{V}}_j^\pm(y;\lambda_*)\right) \qquad (6.34)$$

for some λ_* on the ray from 0 to λ, where, recall, $\left(\frac{\partial}{\partial\lambda}\right)\tilde{\mathbb{V}}_j^\pm(y;\cdot)$ and $\left(\frac{\partial}{\partial\lambda}\right)^2\tilde{\mathbb{V}}_j^\pm(y;\cdot)$ are uniformly bounded in $L^\infty[0,\pm\infty]$ for $\lambda \in \Omega \cap \{\lambda : |\lambda| < \delta\}$. Together with the choice $\tilde V_j^\pm(y;0) \equiv$ constant, this immediately gives the first bound (6.25).

Applying now the bound (6.31) with $k = 1$, we may expand the second coordinate of (6.34) as

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}\right)\tilde W_j^\pm(y;\lambda)\\
= \;& e^{-\mu_j^\pm(\lambda)y}\Big(-\mu_j^\pm(0)\tilde V_j^\pm(y;0) + \tilde V_j^{\pm\prime}(y;0)\\
&-\lambda\left(\left(\frac{\partial}{\partial\lambda}\right)(\mu_j^\pm\tilde V_j^\pm)(0) + \mathbf{O}(e^{-\alpha|y|})\right) + \mathbf{O}(\lambda^2)\Big)\\
= \;& e^{-\mu_j^\pm(\lambda)y}\left(-\lambda\left(\left(\frac{\partial}{\partial\lambda}\right)\mu_j^\pm(0)\tilde V_j^\pm(0) + \mathbf{O}(e^{-\alpha|y|})\right) + \mathbf{O}(\lambda^2)\right),
\end{aligned}$$

and subtracting off the corresponding Taylor expansion

$$\begin{aligned}
&\left(\frac{\partial}{\partial y}\right)\left(e^{-\mu_j^\pm(\lambda)y}\tilde V_j^\pm(y;0)\right)\\
= \;& -\mu_j^\pm(\lambda)e^{-\mu_j^\pm(\lambda)y}\tilde V_j^\pm(y;0)\\
= \;& e^{-\mu_j^\pm(\lambda)y}\left(-\mu_j^\pm(0)\tilde V_j^\pm(y;0) - \lambda\left(\frac{\partial}{\partial\lambda}\right)\mu_j^\pm(0)\tilde V_j^\pm(y;0) + \mathbf{O}(\lambda^2)\right)\\
= \;& e^{-\mu_j^\pm(\lambda)y}\left(-\lambda\left(\frac{\partial}{\partial\lambda}\right)\mu_j^\pm(0)\tilde V_j^\pm(0) + \mathbf{O}(\lambda^2)\right)
\end{aligned}$$

we obtain

$$\lambda \left(\frac{\partial}{\partial y}\right) \tilde{\Theta}_j^{\pm}(y;\lambda) = e^{-\mu_j^{\pm}(\lambda)y} \left(\lambda \mathbf{O}(e^{-\alpha|y|}) + \mathbf{O}(\lambda^2)\right)$$

so

$$\left(\frac{\partial}{\partial y}\right) \tilde{\Theta}_j^{\pm}(y;\lambda) = e^{-\mu_j^{\pm}(\lambda)y} \left(\mathbf{O}(e^{-\alpha|y|}) + \mathbf{O}(\lambda)\right) \le C \left|e^{-\mu_j^{\pm}(\lambda)y}\right| \left(|\lambda| + e^{-\alpha|y|}\right),$$

as claimed. □

We now turn to the estimation of scattering coefficients $m^{\pm}$, $d^{\pm}$ in Corollary 5.5.

Lemma 6.4. *For $|\lambda|$ sufficiently small,*

$$|m^{\pm}|, |d^{\pm}| \le C\lambda^{-1} \tag{6.35}$$

Moreover,

$$\mathrm{Res}_{\lambda=0} m^{\pm} = \mathrm{Res}_{\lambda=0} d^{\pm} \tag{6.36}$$

Proof. Expanding $m^+ = (1,0)(\phi^+,\phi^-)^{-1}\psi^-$ using Cramer's rule, and setting $x = y = 0$ in $\phi^{\pm}$ and ψ^-, we obtain

$$\begin{aligned} m^+ &= (1,0)(\phi^+,\phi^-)^{-1}\psi^- \\ &= \frac{1}{D}(1,0)(\phi^+,\phi^-)^{\mathrm{adj}}\psi^- = \frac{1}{D}c^+ \end{aligned}$$

where $D = \det(\phi^+,\phi^-)$, $c^+ = (1,0)(\phi^+,\phi^-)^{\mathrm{adj}}\psi^-$,

$$m^+ = \frac{\det(\psi^-,\phi^-)}{\det(\phi^+,\phi^-)} = \frac{\det(\psi^-,\phi^-)}{D}$$

so

$$\begin{aligned} c^+ &= Dm^+ = \det(\psi^-,\phi^-) \\ &= \det\left(V_1^-(x;\lambda)e^{\mu_1^-(\lambda)x}, V_2^-(y;\lambda)e^{\mu_2^-(\lambda)y}\right) \\ &= \det\left(\left[V_1^-(\lambda) + \mathbf{O}(e^{-\tilde{\alpha}|x|})\right] e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)x},\right. \\ &\qquad \left. V_2^-(y,0)e^{f'(\bar{u}_-)y} + \lambda \mathbf{O}(e^{-\alpha|y|})\right) \\ &= \det(V_1^-(\lambda) + \mathbf{O}(1), V_2^-(0) + \lambda\mathbf{O}(1)) \\ &= \det\begin{pmatrix} 1 + \mathbf{O}(\lambda) + \mathbf{O}(1) & 1 + \mathbf{O}(\lambda) \\ -\frac{\lambda}{f'(\bar{u}_-)} + \mathbf{O}(\lambda^2) + \mathbf{O}(1) & f'(\bar{u}_-) + \mathbf{O}(\lambda) \end{pmatrix} \\ &= f'(\bar{u}_-) + \frac{\lambda}{f'(\bar{u}_-)} + \mathbf{O}(1) + \mathbf{O}(\lambda) + \mathbf{O}(\lambda^2) + \mathbf{O}(\lambda^3) \\ &= f'(\bar{u}_-) + \mathbf{O}(1) \end{aligned}$$

and

$$
\begin{aligned}
D &= \det(\phi^+, \phi^-) = \det(V_1^+(0;\lambda), V_2^-(0;\lambda)) \\
&= \det(V_1^+(\lambda) + \mathbf{O}(1), V_2^-(\lambda) + \mathbf{O}(1)) \\
&= \det \begin{pmatrix} 1 + \mathbf{O}(\lambda) + \mathbf{O}(1) & 1 + \mathbf{O}(\lambda) + \mathbf{O}(1) \\ f'(\bar{u}_+) + \mathbf{O}(\lambda) + \mathbf{O}(1) & f'(\bar{u}_-) + \mathbf{O}(\lambda) + \mathbf{O}(1) \end{pmatrix} \\
&= f'(\bar{u}_-) - f'(\bar{u}_+) + \mathbf{O}(1) + \mathbf{O}(\lambda) \\
&= \mathbf{O}(\lambda)
\end{aligned}
$$

together we have

$$
m^+ = \frac{c^+}{D} = \frac{f'(\bar{u}_-) + \mathbf{O}(1)}{\mathbf{O}(\lambda)}
$$

or

$$
|m^+| \leq C/\lambda
$$

□

Proposition 6.5. *For $\lambda \in \Omega \cap \{\lambda : |\lambda| < \delta\}$ and δ sufficiently small, the resolvent kernel G_λ has a meromorphic extension onto $\{\lambda : |\lambda| < \delta\}$, which may be decomposed as*

$$
G_\lambda = E_\lambda + S_\lambda + R_\lambda = E_\lambda + S_\lambda + R_\lambda^E + R_\lambda^S \tag{6.37}
$$

For $y \leq 0 \leq x$:

$$
\begin{pmatrix} E_\lambda & E_{\lambda,y} \\ E_{\lambda,x} & E_{\lambda,xy} \end{pmatrix} := C_1 \lambda^{-1} W_1^+(x;0) \tilde{V}_1^-(0) e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right) y} \tag{6.38}
$$

$$
S_\lambda = 0 \tag{6.39}
$$

and

$$
\begin{pmatrix} R_\lambda & R_{\lambda,y} \\ R_{\lambda,x} & R_{\lambda,xy} \end{pmatrix} := e^{-\alpha|x|} e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right) y} \left(\mathbf{O}(\lambda^{-1}) \left(e^{\mathbf{O}(\lambda^3) y} - 1 \right) + \mathbf{O}(1) \right) \tag{6.40}
$$

For $y \leq x \leq 0$:

$$
\begin{pmatrix} E_\lambda & E_{\lambda,y} \\ E_{\lambda,x} & E_{\lambda,xy} \end{pmatrix} := C_2 \lambda^{-1} W_2^-(x;0) \tilde{V}_1^-(0) e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right) y} \tag{6.41}
$$

$$\begin{pmatrix} S_\lambda & S_{\lambda,y} \\ S_{\lambda,x} & S_{\lambda,xy} \end{pmatrix} := V_1^-(0)\tilde{V}_1^-(0)e^{\left(-\frac{\lambda}{f'(\bar{u}_-)}+\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} \tag{6.42}$$

$$\begin{pmatrix} R^E_\lambda & R^E_{\lambda,y} \\ R^E_{\lambda,x} & R^E_{\lambda,xy} \end{pmatrix} := e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}(\lambda^{-1})\left(e^{\mathbf{O}(\lambda^3)y}-1\right)+\mathbf{O}(1)\right) \tag{6.43}$$

$$\begin{pmatrix} R^S_\lambda & R^S_{\lambda,y} \\ R^S_{\lambda,x} & R^S_{\lambda,xy} \end{pmatrix} := e^{\left(-\frac{\lambda}{f'(\bar{u}_-)}+\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)}\left[\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)(x-y)}-1\right)+\mathbf{O}(\lambda)+\mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right] \tag{6.44}$$

In fact, the derivatives of R_λ can have better bounds.

For $y \le 0 \le x$,

$$R_{\lambda,y} = e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)y}-1\right)+\mathbf{O}(\lambda)\right) \tag{6.45}$$

For $y \le x \le 0$,

$$R^E_{\lambda,y} = e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)y}-1\right)+\mathbf{O}(\lambda)\right) \tag{6.46}$$

$$R^S_{\lambda,y} = e^{\left(-\frac{\lambda}{f'(\bar{u}_-)}+\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)}\left[\lambda\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)(x-y)}-1\right)+\mathbf{O}(\lambda)+\lambda\mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right] \tag{6.47}$$

Proof. For $y \le 0 \le x$,

$$
\begin{aligned}
& \begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} \\
= \ & m^+(\lambda)\phi^+(x;\lambda)\tilde{\psi}^-(y;\lambda) \\
= \ & m^+(\lambda)W_1^+(x;\lambda)\tilde{W}_1^-(y;\lambda) \\
= \ & C\lambda^{-1}\left(W_1^+(x;0)+\lambda\mathbf{O}\left(e^{-\alpha|x|}\right)\right)\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\tilde{V}_1^-(0)\right. \\
& \left.+\lambda\mathbf{O}\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\right)\right) \\
= \ & C\lambda^{-1}\left(W_1^+(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\right. \\
& +\lambda\mathbf{O}\left(e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\right) \\
& \left.+\lambda W_1^+(x;0)\mathbf{O}\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\right)\right) \\
= \ & C\lambda^{-1}\left(W_1^+(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\right. \\
& +W_1^+(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y} \\
& -W_1^+(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
& \left.+\lambda\mathbf{O}\left(e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}+\mathbf{O}(\lambda^3)\right)y}\right)\right) \\
= \ & C_1\lambda^{-1}W_1^+(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
& +e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)}-\frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}(\lambda^{-1})\left(e^{\mathbf{O}(\lambda^3)y}-1\right)+\mathbf{O}(1)\right) \\
= \ & \begin{pmatrix} E_\lambda & E_{\lambda,y} \\ E_{\lambda,x} & E_{\lambda,xy} \end{pmatrix}+\begin{pmatrix} R_\lambda & R_{\lambda,y} \\ R_{\lambda,x} & R_{\lambda,xy} \end{pmatrix}
\end{aligned}
$$

Thus we have those representations as claimed.

For $y \le x \le 0$,

$$
\begin{aligned}
&\begin{pmatrix} G_\lambda & G_{\lambda,y} \\ G_{\lambda,x} & G_{\lambda,xy} \end{pmatrix} \\
= \; & d^+(\lambda)\phi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) + \psi^-(x;\lambda)\tilde{\psi}^-(y;\lambda) \\
= \; & d^+(\lambda)W_2^-(x;\lambda)\tilde{W}_1^-(y;\lambda) + W_1^-(x;\lambda)\tilde{W}_1^-(y;\lambda) \\
= \; & C\lambda^{-1}\left(W_2^-(x;0) + \lambda\mathbf{O}\left(e^{-\alpha|x|}\right)\right)\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\tilde{V}_1^-(0)\right. \\
& \left. + \lambda\mathbf{O}\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\right)\right) \\
& + \left(V_1^-(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right)e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)x} \\
& \left[e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\tilde{V}_1^-(0) + \lambda\mathbf{O}\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\right)\right] \\
= \; & C_2\lambda^{-1}W_2^-(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
& + e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}(\lambda^{-1})\left(e^{\mathbf{O}(\lambda^3)y} - 1\right) + \mathbf{O}(1)\right) \\
& + \left(V_1^-(0) + \mathbf{O}(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right)e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)x} \\
& \left[e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\tilde{V}_1^-(0) + \lambda\mathbf{O}\left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y}\right)\right] \\
= \; & C_2\lambda^{-1}W_2^-(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
& + e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}(\lambda^{-1})\left(e^{\mathbf{O}(\lambda^3)y} - 1\right) + \mathbf{O}(1)\right) \\
& + V_1^-(0)\tilde{V}_1^-(0)e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} \\
& + e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)}\left[\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)(x-y)} - 1\right) + \mathbf{O}(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right] \\
= \; & C_2\lambda^{-1}W_2^-(x;0)\tilde{V}_1^-(0)e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
& + V_1^-(0)\tilde{V}_1^-(0)e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} \\
& + e^{-\alpha|x|}e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y}\left(\mathbf{O}(\lambda^{-1})\left(e^{\mathbf{O}(\lambda^3)y} - 1\right) + \mathbf{O}(1)\right) \\
& + e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)}\left[\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)(x-y)} - 1\right) + \mathbf{O}(\lambda) + \mathbf{O}\left(e^{-\tilde{\alpha}|x|}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} E_\lambda & E_{\lambda,y} \\ E_{\lambda,x} & E_{\lambda,xy} \end{pmatrix} + \begin{pmatrix} S_\lambda & S_{\lambda,y} \\ S_{\lambda,x} & S_{\lambda,xy} \end{pmatrix} + \begin{pmatrix} R_\lambda & R_{\lambda,y} \\ R_{\lambda,x} & R_{\lambda,xy} \end{pmatrix} \\
&= \begin{pmatrix} E_\lambda & E_{\lambda,y} \\ E_{\lambda,x} & E_{\lambda,xy} \end{pmatrix} + \begin{pmatrix} S_\lambda & S_{\lambda,y} \\ S_{\lambda,x} & S_{\lambda,xy} \end{pmatrix} + \begin{pmatrix} R^E_\lambda & R^E_{\lambda,y} \\ R^E_{\lambda,x} & R^E_{\lambda,xy} \end{pmatrix} + \begin{pmatrix} R^S_\lambda & R^S_{\lambda,y} \\ R^S_{\lambda,x} & R^S_{\lambda,xy} \end{pmatrix}
\end{aligned}
$$

Then this gives (6.41) – (6.44).

Next we derive the derivative bounds (6.45) – (6.47). We utilize the estimates (6.26).

For $y \le 0 \le x$,

$$
\begin{aligned}
& G_{\lambda,y}(x,y) \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} m^+(\lambda)\phi^+(x;\lambda) \left(\frac{\partial}{\partial y}\right) \tilde{\psi}^-(y;\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} m^+(\lambda)W_1^+(x;\lambda) \left(\frac{\partial}{\partial y}\right) \tilde{W}_1^-(y;\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= C\lambda^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(W_1^+(x;0) + \lambda \mathbf{O}\left(e^{-\alpha|x|}\right)\right) \\
& \quad \left(e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y} \tilde{V}_1^-(0) \left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right) \right. \\
& \quad \left. + C\lambda e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3} + \mathbf{O}(\lambda^3)\right)y} (|\lambda| + e^{-\alpha|y|}) \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

and the remainder term $R_{\lambda,y}(x,y)$ should be

$$
R_{\lambda,y} = e^{-\alpha|x|} e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \left(\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)y} - 1\right) + \mathbf{O}(\lambda)\right)
$$

The $R_{\lambda,y}(x,y)$ estimates for the case $y \le x \le 0$ is similarly derived. □

Remark 6.6. In Proposition 6.5, in fact we can take

$$
W_1^+(x;0) = W_2^-(x;0) = \begin{pmatrix} \bar{u}'(x) \\ \bar{u}''(x) \end{pmatrix}.
$$

Remark 6.7. The The derivative bounds (6.45) – (6.47) is valid only for Lax and over compressive case, it does not hold in under compressive case. See [MaZ].

7 High-frequency bounds

Now we derive the bounds for large $|\lambda|$, on any sector contained in the resolvent set.

Proposition 7.1. *Assuming* $(\mathcal{H})$*, it follows that for some* $C, \beta, R > 0$*, and* $\theta_1, \theta_2 > 0$ *sufficiently small,*

$$|G_\lambda(x, y)| \le C|\lambda|^{-\frac{1}{2}} e^{-\beta^{-\frac{1}{2}}|\lambda|^{\frac{1}{2}}|x-y|}; \tag{7.1}$$

$$|G_{\lambda,x}(x, y)| \le C e^{-\beta^{-\frac{1}{2}}|\lambda|^{\frac{1}{2}}|x-y|}; \tag{7.2}$$

$$|G_{\lambda,y}(x, y)| \le C e^{-\beta^{-\frac{1}{2}}|\lambda|^{\frac{1}{2}}|x-y|}; \tag{7.3}$$

for all $\lambda \in \Omega_\theta \setminus B(0, R)$.
(Here, we may choose any $\beta^{-\frac{1}{2}} < \min_{\lambda \in \Omega_\theta \cap \{\lambda : |\lambda| \ge R\}} \mathrm{Re}(\sqrt{\frac{\lambda}{|\lambda|}})$*.)*

Proof. Setting $\bar{x} = |\lambda|^{\frac{1}{2}} x, \bar{\lambda} = \frac{\lambda}{|\lambda|}, \bar{w}(\bar{x}) = w(\frac{\bar{x}}{|\lambda|^{\frac{1}{2}}}) = w(x)$, we obtain

$$\bar{w}'' = \bar{\lambda}\bar{w} + \mathbf{O}(|\lambda|^{-\frac{1}{2}})(\bar{w} + \bar{w}'), \tag{7.4}$$

or

$$\overline{W}' = \bar{\mathbb{B}}\overline{W} + \mathbf{O}(|\lambda|^{-\frac{1}{2}})\overline{W}, \tag{7.5}$$

where $\overline{W} = (\bar{w}, \bar{w}')^T$, and

$$\bar{\mathbb{B}} := \begin{pmatrix} 0 & 1 \\ \bar{\lambda} & 0 \end{pmatrix}, \bar{\mathbb{B}}' = \mathbf{O}(|\lambda|^{-\frac{1}{2}}), |\bar{\lambda}| = 1. \tag{7.6}$$

It is easily computed that the eigenvalues of $\bar{\mathbb{B}}$ are

$$\bar{\mu} = \mp\sqrt{\bar{\lambda}}, \tag{7.7}$$

We know that there exists some $\beta > 0$, such that

$$\left|\mathrm{Re}\sqrt{\bar{\lambda}}\right| > \beta^{-\frac{1}{2}} \tag{7.8}$$

for all $\lambda \in \Omega_\theta$, hence the stable and unstable subspaces of each $\bar{\mathbb{B}}\bar{x}$ are both of dimension n, and separated by a spectral gap of more than $2\beta^{-\frac{1}{2}}$. Since $\bar{\mathbb{B}}(\lambda, \bar{x})$ varies within a compact set, it follows that there are continuous eigenprojections $P_\pm(\bar{\mathbb{B}})$ taking $\overline{W}$ onto the stable and unstable subspaces, respectively, of $\bar{\mathbb{B}}$, with $|P'_\pm| = \mathbf{O}(|\lambda|^{-\frac{1}{2}})$.
Introducing new coordinates $z_\pm = P_\pm\bar{w}$, we thus obtain a diagonal system

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix}' = \begin{pmatrix} f'(\bar{u}_+) & 0 \\ 0 & f'(\bar{u}_-) \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} + \mathbf{O}\left(|\lambda|^{-\frac{1}{2}}\right) \begin{pmatrix} z_+ \\ z_- \end{pmatrix}, \tag{7.9}$$

We choose β large enough such that

$$\mathrm{Re} f'(\bar{u}_\pm) \lesseqgtr \mp \beta^{-\frac{1}{2}}. \tag{7.10}$$

and hence

$$\frac{|\bar{w}|}{C} \leq |z| \leq C|\bar{w}|. \tag{7.11}$$

From (7.9), we obtain the "energy estimates"

$$\begin{aligned} \langle z_\pm, z_\pm \rangle' &= \langle z_\pm, 2\mathrm{Re} f'(\bar{u}_\pm) z_\pm \rangle + \mathrm{O}\left(|\lambda|^{-\frac{1}{2}}\right)(\langle z_+, z_+\rangle + \langle z_-, z_-\rangle) \quad (7.12) \\ &\lesseqgtr \mp\beta^{-\frac{1}{2}} \langle z_\pm, z_\pm \rangle + \mathrm{O}\left(|\lambda|^{-\frac{1}{2}}\right)(\langle z_+, z_+\rangle + \langle z_-, z_-\rangle) \quad (7.13) \end{aligned}$$

In consequence, the ratios $r_+ := \frac{\langle z_-, z_-\rangle}{\langle z_+, z_+\rangle}$ and $r_- := \frac{\langle z_+, z_+\rangle}{\langle z_-, z_-\rangle}$ satisfy

$$r'_\pm \gtreqless \pm 4\beta^{-\frac{1}{2}} r_\pm \mp C|\lambda|^{-\frac{1}{2}}(1 + r_\pm + r_\pm^2) \tag{7.14}$$

for some $C > 0$. From (7.14) it follows easily that the cones $\mathbb{K}_\mp := \{0 < r_\mp < \frac{\beta^{-\frac{1}{2}}}{C}|\lambda|^{\frac{1}{2}}\}$ are invariant under forward and backward flow, respectively, of (7.9), provided that

$$C|\lambda|^{-\frac{1}{2}}\beta^{\frac{1}{2}} < \frac{4}{3}. \tag{7.15}$$

Since the stable/unstable subspaces of $\begin{pmatrix} f'(\bar{u}_+) & 0 \\ 0 & f'(\bar{u}_-) \end{pmatrix}$ at $x = \pm\infty$ are precisely $\{z_\pm = 0\}$, we have that the stable/unstable subspaces of $\begin{pmatrix} f'(\bar{u}_+) & 0 \\ 0 & f'(\bar{u}_-) \end{pmatrix} + \mathbf{O}\left(|\lambda|^{-\frac{1}{2}}\right)$ at $x = \pm\infty$ lie within the respective cones $\mathbb{K}_\pm$, provided $|\lambda|$ is sufficiently large. It follows that the stable/unstable manifolds of solutions of (7.9) lie within $\mathbb{K}_\pm$ for all x.
Plugging this information back into (7.12), we find that

$$(|z_\pm|^2)' \lesseqgtr \mp 2\tilde{\beta}^{-\frac{1}{2}}|z_\pm|^2 \tag{7.16}$$

for any solution $(z_+, z_-)^T$ decaying at $x = \pm\infty$, hence

$$\frac{|z_+(x)|}{|z_-(y)|} \leq e^{-\tilde{\beta}^{-\frac{1}{2}}|x-y|},$$

where $0 < \tilde{\beta} < \beta$, and thus

$$\frac{|z(x)|}{|z(y)|} \leq C_1 e^{-\tilde{\beta}^{-\frac{1}{2}}|x-y|}, \tag{7.17}$$

for any $x \lesseqgtr y$, provided $|\lambda|$ is sufficiently large. This gives

$$\frac{\overline{W}(x)}{\overline{W}(y)} \leq C_1 C^2 e^{-\tilde{\beta}^{-\frac{1}{2}}|x-y|}, \tag{7.18}$$

where C is as in (7.11). Further, untangling intermediate coordinate changes, we find that

Proposition 7.2. ($\mathbb{K}$) *The stable/unstable manifolds of solutions of (7.5) lie within angle* $\mathbf{O}\left(|\lambda|^{-\frac{1}{2}}\right)$ *of the stable/unstable subspaces of* $\bar{\mathbb{B}}(x)$.

Now, recall the coordinate-free representation of the Green's function as

$$\begin{pmatrix} G_\lambda \\ G_{\lambda,x} \end{pmatrix} = \mathcal{F}^{y\to x}\Pi_+(y)\begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix}.$$

Translating the bound (7.18) back to the original system (2.1), we obtain

$$|\mathcal{F}^{y\to x}| \leq C_1 C^2 e^{-\tilde{\beta}|\lambda|^{\frac{1}{2}}|x-y|}, \tag{7.19}$$

Likewise, the projection Π_+ can be related to its counterparts $\bar{\Pi}_+$ for the rescaled system by the factorization

$$\Pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & |\lambda|^{\frac{1}{2}} \end{pmatrix} \bar{\Pi}_+ \begin{pmatrix} 1 & 0 \\ 0 & |\lambda|^{-\frac{1}{2}} \end{pmatrix},$$

and similarly for $\tilde{\Pi}_-$. Since the stable/unstable manifolds stay separated, by Proposition ($\mathbb{K}$), and $\bar{\lambda}$ varies within a compact set, the projections $\bar{\Pi}_+$ and $\tilde{\bar{\Pi}}_-$ are uniformly bounded. Thus, we have

$$\begin{aligned} \Pi_+(y)\begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & |\lambda|^{\frac{1}{2}} \end{pmatrix} \mathbf{O}(1) \begin{pmatrix} 1 & 0 \\ 0 & |\lambda|^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 0 \\ B^{-1}(y) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{O}\left(|\lambda|^{-\frac{1}{2}}\right) \\ \mathbf{O}(1) \end{pmatrix}. \end{aligned}$$

Combining with (7.19), and recalling that $0 < \tilde{\beta} < \beta$ was arbitrary in (7.8), we obtain the claimed bounds on $|G_\lambda|$ and $|G_{\lambda,x}|$. The bound on $|G_{\lambda,y}|$ follows by symmetric argument applied to the adjoint operator L^*, or, equivalently, using the symmetric representation

$$\begin{pmatrix} G_\lambda & G_{\lambda,y} \end{pmatrix} = \begin{pmatrix} 0 & B^{-1}(y) \end{pmatrix} \tilde{\Pi}_-(x)\tilde{\mathcal{F}}^{x\to y},$$

where $\tilde{\mathcal{F}}^{x\to y}$ denotes the flow of the adjoint eigenvalue equation. □

8 Pointwise Green's function bounds

In this section, our starting point is the representation in [MaZ] and [ZH],

$$G(x,t;y) = \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}G_\lambda(x,y)d\lambda \quad (8.1)$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\eta-iT}^{\eta+iT} e^{\lambda t}G_\lambda(x,y)d\lambda \quad (8.2)$$

which is valid for η sufficiently large.

Proposition 8.1. *Under assumption* $(\mathcal{H})$*, the Green's function G(x,t;y) associated with the linearized evolution equation* (1.5) *may be decomposed as*

$$G(x,t;y) = E(x,t;y) + S(x,t;y) + R(x,t;y) \quad (8.3)$$

where for $y \leq 0$ *:*

$$E(x,t;y) = C\bar{u}'(x)\left(\text{errfn}\left(\frac{y+f'(\bar{u}_-)t}{\sqrt{4t}}\right) - \text{errfn}\left(\frac{y-f'(\bar{u}_-)t}{\sqrt{4t}}\right)\right), \quad (8.4)$$

and

$$S(x,t;y) = \chi_{\{t\geq 1\}}(4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}}\left(\frac{e^{-x}}{e^x+e^{-x}}\right), \quad (8.5)$$

$$R(x,t;y) = \mathbf{O}\left(e^{-\eta t}e^{-\frac{|x-y|^2}{Mt}}\right)+\mathbf{O}\left((t+1)^{-\frac{1}{2}}e^{-\eta x^+}+e^{-\eta|x|}\right)t^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}} \quad (8.6)$$

$$R_y(x,t;y) = \mathbf{O}\left(e^{-\eta t}e^{-\frac{|x-y|^2}{Mt}}\right)+\mathbf{O}\left((t+1)^{-\frac{1}{2}}e^{-\eta x^+}+e^{-\eta|x|}\right)t^{-1}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}} \quad (8.7)$$

Proof. **Case I.** $\frac{|x-y|}{t}$ **large.** We first treat the trivial case that $\frac{|x-y|}{t} \geq S$, S sufficiently large, the regime in which standard short-time parabolic theory applies. Set

$$\bar{\alpha} := \frac{|x-y|}{2\beta t}, \quad R := \beta\bar{\alpha}^2, \quad (8.8)$$

where β is as in Proposition 7.1, and consider again the following representation of G, that is

$$G(x,t;y) = \frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} e^{\lambda t}G_\lambda(x,y)d\lambda, \quad (8.9)$$

where $\Gamma_1 := \partial B(0,R) \cap \bar{\Omega}_\theta$ and $\Gamma_2 := \partial\Omega_\theta \backslash B(0,R)$. Note that the intersection of Γ with the real axis is $\lambda_{\min} = R = \beta\bar{\alpha}^2$. By the large $|\lambda|$ estimates of Proposition 7.1, we have for all $\lambda \in \Gamma_1 \cup \Gamma_2$ that

$$|G_\lambda(x,y)| \le C|\lambda|^{-1/2} e^{-\beta^{-\frac{1}{2}}|\lambda|^{\frac{1}{2}}|x-y|} \tag{8.10}$$

Further, we have

$$\begin{aligned} \mathrm{Re}\lambda &\le R(1-\eta\omega^2), \quad \lambda \in \Gamma_1, && (8.11)\\ \mathrm{Re}\lambda &\le \mathrm{Re}\lambda_0 - \eta(|\mathrm{Im}\lambda| - |\mathrm{Im}\lambda_0|), \quad \lambda \in \Gamma_2, && (8.12) \end{aligned}$$

for R sufficiently large, where ω is the argument of λ and λ_0 and λ_0^* are the two points of intersection of Γ_1 and Γ_2, for some $\eta > 0$ independent of $\bar{\alpha}$.

Combining (8.10),(8.11) and (8.8), we obtain

$$\begin{aligned} \left| \int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y) d\lambda \right| &\le \int_{\Gamma_1} C|\lambda|^{-\frac{1}{2}} e^{(\mathrm{Re}\lambda)t - \beta^{-\frac{1}{2}}|\lambda^{\frac{1}{2}}||x-y|} d\lambda \\ &\le Ce^{-\beta\bar{\alpha}^2 t} \int_{-M}^{+M} R^{-\frac{1}{2}} e^{-\beta R\eta\omega^2 t} R d\omega \\ &\le Ct^{-\frac{1}{2}} e^{-\beta\bar{\alpha}^2 t}. \end{aligned}$$

Likewise,

$$\begin{aligned} \left| \int_{\Gamma_2} e^{\lambda t} G_\lambda(x,y) d\lambda \right| &\le \int_{\Gamma_2} C|\lambda|^{-\frac{1}{2}} C e^{(\mathrm{Re}\lambda)t - \beta^{-\frac{1}{2}}|\lambda|^{\frac{1}{2}}||x-y|} d\lambda \\ &\le Ce^{(\mathrm{Re}\lambda_0)t - \beta^{-\frac{1}{2}}|\lambda_0|^{\frac{1}{2}}|x-y|} \int_{\Gamma_2} |\lambda|^{-\frac{1}{2}} e^{(\mathrm{Re}\lambda - \mathrm{Re}\lambda_0)t} |d\lambda| \\ &\le Ce^{-\beta\bar{\alpha}^2 t} \int_{\Gamma_2} |\mathrm{Im}\lambda|^{-\frac{1}{2}} e^{-\eta(|\mathrm{Im}\lambda| - |\mathrm{Im}\lambda_0|)t} |d\mathrm{Im}\lambda| \\ &\le Ct^{-\frac{1}{2}} e^{-\beta\bar{\alpha}^2 t}. \end{aligned}$$

Combining these last two estimates, and recalling (8.8), we have

$$|G(x,t;y)| \le Ct^{-\frac{1}{2}} e^{-\frac{\beta\bar{\alpha}^2 t}{2}} e^{-\frac{(x-y)^2}{8\beta t}} \le Ct^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{(x-y)^2}{8\beta t}}, \tag{8.13}$$

for $\eta > 0$ independent of $\bar{\alpha}$. Observing that

$$\frac{|x-y-at|}{2t} \le \frac{|x-y|}{t} \le \frac{2|x-y-at|}{t}$$

for any bounded a, for $|x-y|/t$ sufficiently large, we find that $|G|$ can be absorbed in the residual term $\mathbf{O}\left(e^{-\eta t} e^{-\frac{|x-y|^2}{Mt}}\right)$ for $t \ge \epsilon$, any $\epsilon > 0$, and in the residual term $\mathbf{O}\left((t+1)^{-\frac{1}{2}}e^{-\eta x^+}t^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar u_-)t)^2}{Mt}}\right)$ for t small.

Case II. $\frac{|x-y|}{t}$ bounded. We now turn to the critical case where $\frac{|x-y|}{t}\le S$ for some fixed S. In this regime, note that any contribution of order $e^{\theta t}, \theta>0$, may be absorbed in the residual term R; we shall use this observation repeatedly. We begin by converting contour integral (8.1) into a more convenient form decomposing high, intermediate, and low frequency contributions.

Lemma 8.2. *If $(\mathcal{H})$ holds, we can use the following decomposition:*

$$G(x,t;y) = \mathbf{I}+\mathbf{II} \tag{8.14}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_2} e^{\lambda t}G_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda t}G_\lambda(x,y)d\lambda \tag{8.15}$$

$$\Gamma' := [-\eta_1 - iR, \eta - iR]\cup[\eta - iR, \eta+iR]\cup[\eta+iR, -\eta_1+iR], \tag{8.16}$$

$$\Gamma_2 := \partial\Omega_\theta\setminus\Omega \tag{8.17}$$

with Ω_θ as defined in section 5, for any $\eta>0$ such that (8.1) holds, R sufficiently large, and $\eta_1>0$ sufficiently small such that $\Omega\setminus B(0,r)$ is compactly contained in the set of consistent splitting Λ for some small $r>0$ to be chosen later, where $\Omega := \{\lambda : -\eta_1 \le \mathrm{Re}\lambda\}$.

Lemma 8.3. *The term* **II** *in Lemma 8.2 may be further decomposed as*

$$\mathbf{II} = \tilde{\mathbf{II}} + \mathbf{III} \tag{8.18}$$

$$= \frac{1}{2\pi i}\left(\int_{-\eta_1-iR}^{-\eta_1-i\frac{r}{2}} + \int_{-\eta_1+i\frac{r}{2}}^{-\eta_1+iR}\right)e^{\lambda t}G_\lambda(x,y)d\lambda \tag{8.19}$$

$$+\frac{1}{2\pi i}\int_{\tilde\Gamma} e^{\lambda t}G_\lambda(x,y)d\lambda \tag{8.20}$$

and

$$\tilde\Gamma := [-\eta_1 - i\frac{r}{2}, \eta - i\frac{r}{2}]\cup[\eta - i\frac{r}{2}, \eta + i\frac{r}{2}]\cup[\eta + i\frac{r}{2}, -\eta_1 + i\frac{r}{2}], \tag{8.21}$$

for any $\eta, r>0$, and η_1 sufficiently small with respect to r.

The proofs of these two lemmas are trivial. We are going to estimate terms $\mathbf{I}, \tilde{\mathbf{II}}$ and $\mathbf{III}$ respectively.

The term $\mathbf{I}$ may be estimated exactly as was term $\int_{\Gamma_2} e^{\lambda t}G_\lambda(x,y)d\lambda$ in the large $\frac{|x-y|}{t}$ case (Case I.), to obtain contribution $\mathbf{O}\left(t^{-\frac{1}{2}}e^{-\eta_1 t}\right)$ absorbable again in the residual term $\mathbf{O}\left(e^{-\eta t}e^{-\frac{|x-y|^2}{Mt}}\right)$ for $t\ge\epsilon$, any $\epsilon>0$, and in the residual term $\mathbf{O}\left((t+1)^{-\frac{1}{2}}e^{-\eta x^+}t^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}\right)$ for t small. To estimate the term $\tilde{\mathbf{II}}$, we use the fact that $|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}$ for λ on any compact subset K of $\rho(L)\cap\Lambda$, where $C>0$ and $\eta>0$ depend only on K, L.

$$\begin{aligned}
\left|\tilde{\mathbf{II}}\right| &\le \frac{1}{2\pi}\left(\left|\int_{-R}^{-\frac{r}{2}} e^{(-\eta_1+i\xi)t}e^{-\eta|x-y|}d\xi + \int_{\frac{r}{2}}^{R} e^{(-\eta_1+i\xi)t}e^{-\eta|x-y|}d\xi\right|\right)\\
&= \frac{1}{2\pi}e^{-\eta_1 t}e^{-\eta|x-y|}\left|\int_{-R}^{-\frac{r}{2}} e^{i\xi t}d\xi + \int_{\frac{r}{2}}^{R} e^{i\xi t}d\xi\right|\\
&= \frac{1}{2\pi}e^{-\eta_1 t}e^{-\eta|x-y|}t^{-1}\left|e^{-i\frac{rt}{2}} - e^{-iRt} + e^{iRt} - e^{i\frac{rt}{2}}\right|\\
&\le \frac{2}{\pi}t^{-1}e^{-\eta_1 t}e^{-\eta|x-y|}
\end{aligned}$$

Thus $\tilde{\mathbf{II}}$ can be absorbed in the residual term R.

It remains to estimate the low frequency term $\mathbf{III} = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}G_\lambda(x,y)d\lambda$.

Case. $t \le 1$**.** First observe that estimates in the short-time regime $t\le 1$ are trivial, since then $|e^{\lambda t}G_\lambda(x,y)|$ is uniformly bounded on the compact set $\tilde{\Gamma}$, and we have $|G(x,t;y)| \le C \le e^{-\theta t}$ for $\theta > 0$ sufficiently small. But, likewise, E and S are uniformly bounded in this regime, hence time-exponentially decaying. As observed previously, all such terms are negligible, begin absorbable in the error term R. Thus, we may add $E+S$ and subtract G to obtain the result.

Case. $t \ge 1$**.** Next, consider the critical (long-time) regime $t\ge 1$. For definiteness, take $y \le x \le 0$; the other two cases are similar. Decomposing,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}G_\lambda(x,y)d\lambda &= \frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}E_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}S_\lambda(x,y)d\lambda\\
&\quad+\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}R_\lambda(x,y)d\lambda
\end{aligned}$$

with E_λ, S_λ and R_λ as defined in Proposition 6.5, we consider in turn each of the three terms on the right-hand side.

The E_λ *term.* Let us first consider the dominant term

$$\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}E_\lambda(x,y)d\lambda \tag{8.22}$$

which by (6.38) is given by

$$C_1\bar{u}'(x)\Xi(x,t;y), \tag{8.23}$$

where

$$\Xi(x,t;y) := \frac{1}{2\pi i}\int_{\tilde\Gamma} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda. \tag{8.24}$$

Using Cauchy's theorem,

$$\begin{aligned} &\frac{1}{2\pi i}\left(\int_{\tilde\Gamma} + \int_{-\eta_1+i\frac{r}{2}}^{-\eta_1-i\frac{r}{2}}\right) \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ = \;& \mathrm{Res}_{\lambda=0} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} \end{aligned}$$

thus we may move the contour $\tilde\Gamma$ to obtain

$$\begin{aligned} \Xi(x,t;y) &= \frac{1}{2\pi i}\int_{\tilde\Gamma} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ &= -\frac{1}{2\pi i}\int_{-\eta_1+i\frac{r}{2}}^{-\eta_1-i\frac{r}{2}} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ &\quad + \mathrm{Res}_{\lambda=0} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} \\ &= \frac{1}{2\pi i}\left(\int_{-\eta_1-i\frac{r}{2}}^{-i\frac{r}{2}} + \int_{i\frac{r}{2}}^{-\eta_1+i\frac{r}{2}}\right) \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ &\quad + \frac{1}{2\pi i}\left(\int_{-i\frac{r}{2}}^{-i\delta} + \int_{i\delta}^{i\frac{r}{2}}\right) \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{\gamma} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda \\ &\quad + \mathrm{Res}_{\lambda=0} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} \end{aligned}$$

where γ is the left half circle $\gamma := \{\delta e^{i\theta} : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$, for some $\delta > 0$. Notice that

$$\mathrm{Res}_{\lambda=0} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} = \lim_{\lambda\to 0} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} = 1.$$

and

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\gamma} \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar u_-)} - \frac{\lambda^2}{(f'(\bar u_-))^3}\right)y} d\lambda = -\frac{1}{2}$$

thus sending $\delta \to 0$ in the above calculations gives us

$$\begin{aligned}
\Xi(x,t;y) &= \frac{1}{2\pi}\mathrm{P.V.}\int_{-\frac{r}{2}}^{\frac{r}{2}} (i\xi)^{-1} e^{i\xi t} e^{\left(\frac{i\xi}{f'(\bar{u}_-)} + \frac{\xi^2}{(f'(\bar{u}_-))^3}\right)y} d\xi \\
&\quad + \frac{1}{2\pi i}\left(\int_{-\eta_1 - i\frac{r}{2}}^{-i\frac{r}{2}} + \int_{i\frac{r}{2}}^{-\eta_1 + i\frac{r}{2}}\right) \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} d\lambda \\
&\quad + \frac{1}{2}\mathrm{Res}_{\lambda=0}\lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} \\
&= \left(\frac{1}{2\pi}\mathrm{P.V.}\int_{-\infty}^{+\infty} (i\xi)^{-1} e^{i\xi\left(t + \frac{y}{f'(\bar{u}_-)}\right)} e^{\xi^2 \frac{y}{(f'(\bar{u}_-))^3}} d\xi + \frac{1}{2}\right) \\
&\quad - \frac{1}{2\pi}\left(\int_{-\infty}^{-\frac{r}{2}} + \int_{\frac{r}{2}}^{+\infty}\right) (i\xi)^{-1} e^{i\xi\left(t + \frac{y}{f'(\bar{u}_-)}\right)} e^{\xi^2 \frac{y}{(f'(\bar{u}_-))^3}} d\xi \\
&\quad + \frac{1}{2\pi i}\left(\int_{-\eta_1 - i\frac{r}{2}}^{-i\frac{r}{2}} + \int_{i\frac{r}{2}}^{-\eta_1 + i\frac{r}{2}}\right) \lambda^{-1} e^{\lambda t} e^{\left(\frac{\lambda}{f'(\bar{u}_-)} - \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)y} d\lambda
\end{aligned}$$

The first term in the above equality may be explicitly evaluated to give

$$\mathrm{errfn}\left(\frac{y + f'(\bar{u}_-)t}{\sqrt{4\left|\frac{y}{f'(\bar{u}_-)}\right|}}\right), \tag{8.25}$$

where

$$\mathrm{errfn}(z) := \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z} e^{-y^2} dy, \tag{8.26}$$

whereas the second and third terms are clearly time-exponentially small for $t \le C|y|$ and η_1 sufficiently small relative to r. In the trivial case $t \ge C|y|, C > 0$ sufficiently large, we can simply move the contour to $[-\eta_1 - i\frac{r}{2}, -\eta_1 + i\frac{r}{2}]$ to obtain a complete residue of 1 plus a time-exponentially small error corresponding to the shifted contour integral, which result again agrees with (8.25) up to a time-exponentially small error.

Expression (8.25) may be rewritten as

$$\mathrm{errfn}\left(\frac{y + f'(\bar{u}_-)t}{\sqrt{4t}}\right), \tag{8.27}$$

plus error

$$\begin{aligned}
&\operatorname{errfn}\left(\frac{y+f'(\bar u_-)t}{\sqrt{4\left|\frac{y}{f'(\bar u_-)}\right|}}\right)-\operatorname{errfn}\left(\frac{y+f'(\bar u_-)t}{\sqrt{4t}}\right)\\
=\;&\operatorname{errfn}'\left(\frac{y+f'(\bar u_-)t}{\sqrt{4t}}\right)\left(-2(y+f'(\bar u_-)t)^2(4t)^{-\frac{3}{2}}\right)\\
&=\mathrm{O}(t^{-1}e^{\frac{(y+f'(\bar u_-)t)^2}{Mt}}),
\end{aligned} \tag{8.28}$$

for $M>0$ sufficiently large, and similarly for $x-$ and $y-$derivatives. Multiplying by

$$C_1\bar u'(x)=\mathrm{O}\left(e^{-\alpha|x|}\right)$$

we find that term (8.27) gives contribution

$$C_1\bar u'(x)\operatorname{errfn}\left(\frac{y+f'(\bar u_-)t}{\sqrt{4t}}\right) \tag{8.29}$$

whereas term (8.28) gives a contribution absorbable in R.

Finally, observing that

$$C_1\bar u'(x)\operatorname{errfn}\left(\frac{y-f'(\bar u_-)t}{\sqrt{4t}}\right) \tag{8.30}$$

is time-exponentially small for $t\geq 1$, since $f'(\bar u_-)>0, y<0$, and $|\bar u'(x)|\leq Ce^{-\alpha|x|}$, we may subtract and add this term to (8.29) to obtain a total of

$$E(x,t;y)=C\bar u'(x)\left(\operatorname{errfn}\left(\frac{y+f'(\bar u_-)t}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{y-f'(\bar u_-)t}{\sqrt{4t}}\right)\right),$$

plus terms absorbable in R.

The S_λ term. Next, we consider the second-order term

$$\frac{1}{2\pi i}\int_{\tilde\Gamma}e^{\lambda t}S_\lambda(x,y)d\lambda \tag{8.31}$$

which by (6.42), is given by

$$C_2\Xi'(x,t;y) \tag{8.32}$$

where

$$\Xi':=\frac{1}{2\pi i}\int_{\tilde\Gamma}e^{\lambda t}e^{\left(-\frac{\lambda}{f'(\bar u_-)}+\frac{\lambda^2}{(f'(\bar u_-))^3}\right)(x-y)}d\lambda \tag{8.33}$$

Similarly as in the treatment of the E_λ term, just above, by deforming the contour $\tilde{\Gamma}$ to

$$\Gamma'' := [-\eta_1 - i\frac{r}{2}, -i\frac{r}{2}] \cup [-i\frac{r}{2}, +i\frac{r}{2}] \cup [+i\frac{r}{2}, -\eta_1 + i\frac{r}{2}], \tag{8.34}$$

these may be transformed, neglecting time-exponentially decaying terms, to the elementary Fourier integrals

$$\frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{+\infty} e^{i\xi\left(t-\frac{x-y}{f'(\bar{u}_-)}\right)} e^{\xi^2\left(-\frac{1}{(f'(\bar{u}_-))^3}\right)(x-y)} d\xi$$
$$= (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}}$$

Noting that for $t \geq 1, y \leq x \leq 0$, there is

$$\left|(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \left(1 - \frac{e^{-x}}{e^x + e^{-x}}\right)\right|$$
$$\leq (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} e^{-\alpha|x|}$$

for some $\alpha > 0$, so is absorbable in error term R, we find that the total contribution of this term, neglecting terms aborbable in R, is

$$S(x,t;y) = \chi_{\{t\geq 1\}}(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \left(\frac{e^{-x}}{e^x + e^{-x}}\right)$$

.

The R_λ term. Finally, we briefly discuss the estimation of error term

$$\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t} R_\lambda(x,y) d\lambda \tag{8.35}$$

We can decompose the above integral into sum of integrals involving various terms of R_λ^E and R_λ^S given in (6.43) and (6.44). By expanding the term $\mathbf{O}\left(e^{\mathbf{O}(\lambda^3)(x-y)} - 1\right)$, we get contour integrals of the form

$$\frac{1}{2\pi i}\int_{\tilde{\Gamma}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda \tag{8.36}$$

It may be deformed to contour

$$\Gamma''' := [-\eta_1 - i\frac{r}{2}, \eta_* - i\frac{r}{2}] \cup [\eta_* - i\frac{r}{2}, \eta_* + i\frac{r}{2}] \cup [\eta_* + i\frac{r}{2}, -\eta_1 + i\frac{r}{2}] \tag{8.37}$$

where the saddle-point η_* is defined as

$$\eta_*(x,y,t) := \begin{cases} \frac{\bar{\alpha}}{p}, & \text{if} \quad \left|\frac{\bar{\alpha}}{p}\right| \leq \varepsilon; \\ \\ \pm\varepsilon, & \text{if} \quad \frac{\bar{\alpha}}{p} \gtrless \pm\varepsilon, \end{cases} \tag{8.38}$$

with

$$\bar{\alpha} := \frac{x - y - f'(\bar{u}_-)t}{2t}, p := \frac{x-y}{(f'(\bar{u}_-))^2 t} > 0, \tag{8.39}$$

so the integral (8.36) may be rewritten as

$$\begin{aligned} & \frac{1}{2\pi i}\int_{-\eta_1 - i\frac{r}{2}}^{\eta_* - i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda \\ + \quad & \frac{1}{2\pi i}\int_{\eta_* - i\frac{r}{2}}^{\eta_* + i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda \\ + \quad & \frac{1}{2\pi i}\int_{\eta_* + i\frac{r}{2}}^{-\eta_1 + i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda \end{aligned}$$

with a bit of computation we can show that the main contribution lies along the central vertical portion $[\eta_* - i\frac{r}{2}, \eta_* + i\frac{r}{2}]$ of the contour Γ''':

$$\frac{1}{2\pi i}\int_{\eta_* - i\frac{r}{2}}^{\eta_* + i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda \tag{8.40}$$

Now we estimate (8.40), set $\lambda = \eta_* + i\xi$ where $-\frac{r}{2} \le \xi \le \frac{r}{2}$. Because

$$\begin{aligned}
& \mathrm{Re}\left(\lambda t + \left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)\right) \\
= & \mathrm{Re}\left(-\lambda \frac{2t}{f'(\bar{u}_-)}\left(\frac{x-y-f'(\bar{u}_-)t}{2t}\right) + \frac{\lambda^2 t}{f'(\bar{u}_-)}\frac{(x-y)}{(f'(\bar{u}_-))^2 t}\right) \\
= & \mathrm{Re}\left(-\frac{2\lambda t}{f'(\bar{u}_-)}\bar{\alpha} + \frac{\lambda^2 t}{f'(\bar{u}_-)}p\right) \\
= & \mathrm{Re}\left(-\frac{t}{f'(\bar{u}_-)}\left(2\bar{\alpha}\lambda - p\lambda^2\right)\right) \\
= & -\frac{t}{f'(\bar{u}_-)}\left(2\bar{\alpha}\mathrm{Re}(\lambda) - p\mathrm{Re}(\lambda^2)\right) \\
= & -\frac{t}{f'(\bar{u}_-)}\left(2\bar{\alpha}\mathrm{Re}(\eta_* + i\xi) - p\mathrm{Re}(\eta_* + i\xi)^2\right) \\
= & -\frac{t}{f'(\bar{u}_-)}\left(2\bar{\alpha}\eta_* - p\eta_*^2 + p\xi^2\right) \\
= & -\frac{t}{f'(\bar{u}_-)}\frac{\bar{\alpha}^2}{p} - \frac{tp}{f'(\bar{u}_-)}\xi^2
\end{aligned}$$

and $|\lambda|^q = |\eta_* + i\xi|^q \le \mathbf{O}(|\eta_*|^q + |\xi|^q)$, we have

$$\begin{aligned}
& \left|\frac{1}{2\pi i}\int_{\eta_* - i\frac{r}{2}}^{\eta_* + i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda\right| \\
\le & e^{-\frac{t}{f'(\bar{u}_-)}\frac{\bar{\alpha}^2}{p}} \int_{-\frac{r}{2}}^{\frac{r}{2}} \mathbf{O}(|\eta_*|^q + |\xi|^q) e^{-\frac{tp}{f'(\bar{u}_-)}\xi^2} d\xi \\
\le & e^{-\frac{(f'(\bar{u}_-))^2}{x-y}\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \int_{-\infty}^{\infty} \mathbf{O}(|\eta_*|^q + |\xi|^q) e^{-\frac{p}{f'(\bar{u}_-)}\xi^2 t} d\xi \\
\le & \mathbf{O}\left(t^{-\frac{q+1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}\right)
\end{aligned}$$

if $\left|\frac{\bar{\alpha}}{p}\right| \le \varepsilon$, and

$$\begin{aligned}
& \left|\frac{1}{2\pi i}\int_{\eta_* - i\frac{r}{2}}^{\eta_* + i\frac{r}{2}} e^{\lambda t}\lambda^q e^{\left(-\frac{\lambda}{f'(\bar{u}_-)} + \frac{\lambda^2}{(f'(\bar{u}_-))^3}\right)(x-y)} d\lambda\right| \\
\le & e^{-\frac{\varepsilon t}{M}} \int_{-\infty}^{\infty} \mathbf{O}(|\eta_*|^q + |\xi|^q) e^{-\frac{p}{f'(\bar{u}_-)}\xi^2 t} d\xi \\
\le & \mathbf{O}\left(t^{-\frac{q+1}{2}} e^{-\eta t}\right)
\end{aligned}$$

if $\left|\frac{\bar{\alpha}}{p}\right| \geq \varepsilon$. Combining these estimates, we get the bound in (8.6). $\square$

Remark 8.4. The derivation of (8.25). We are going to evaluate the integral,

$$\frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{+\infty}(i\xi)^{-1}e^{i\xi\left(t+\frac{y}{f'(\bar{u}_-)}\right)}e^{\xi^2\frac{y}{(f'(\bar{u}_-))^3}}d\xi$$

We make a change of variable $\zeta = \sqrt{\frac{-y}{(f'(\bar{u}_-))^3}}\xi$, then $\frac{y}{(f'(\bar{u}_-))^3}\xi^2 = -\zeta^2$ and $d\xi = \frac{1}{\sqrt{\frac{-y}{(f'(\bar{u}_-))^3}}}d\zeta$. So the integral becomes

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\frac{-y}{(f'(\bar{u}_-))^3}}}{i\zeta}e^{i\left(t+\frac{y}{f'(\bar{u}_-)}\right)\frac{1}{\sqrt{\frac{-y}{(f'(\bar{u}_-))^3}}}\zeta}e^{-\zeta^2}\frac{1}{\sqrt{\frac{-y}{(f'(\bar{u}_-))^3}}}d\zeta \\
= & \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{i\zeta}e^{i\left(\frac{y+f'(\bar{u}_-)t}{\sqrt{-\frac{y}{f'(\bar{u}_-)}}}\right)\zeta}e^{-\zeta^2}d\zeta \\
= & \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{-\zeta^2}}{i\zeta}e^{i\left(\frac{y+f'(\bar{u}_-)t}{\sqrt{-\frac{y}{f'(\bar{u}_-)}}}\right)\zeta}d\zeta \\
= & \frac{1}{\sqrt{2\pi}}\left(\mathcal{F}_\zeta^{-1}f(\zeta)\right)\left(\frac{y+f'(\bar{u}_-)t}{\sqrt{-\frac{y}{f'(\bar{u}_-)}}}\right) \\
= & \frac{1}{\sqrt{2\pi}}\left(\mathcal{F}_\zeta^{-1}f(\zeta)\right)(\tau) := \frac{1}{\sqrt{2\pi}}g(\tau)
\end{aligned}$$

where $f(\zeta) = \frac{e^{-\zeta^2}}{i\zeta}$, $\tau = \frac{y+f'(\bar{u}_-)t}{\sqrt{-\frac{y}{f'(\bar{u}_-)}}}$ and $g(\tau) = \left(\mathcal{F}_\zeta^{-1}f(\zeta)\right)(\tau)$. By the inverse Fourier transform formulae, we have

$$f(\zeta) = \left(\mathcal{F}_\tau g(\tau)\right)(\zeta)$$

and

$$i\zeta f(\zeta) = \left(\mathcal{F}_\tau g'(\tau)\right)(\zeta) = e^{-\zeta^2}$$

so

$$\begin{aligned} g'(\tau) &= \left(\mathcal{F}_\zeta^{-1} e^{-\zeta^2}\right)(\tau) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\zeta^2} e^{i\tau\zeta} d\zeta \\ &= \frac{1}{\sqrt{2}} e^{-\frac{\tau^2}{4}} \\ &= \frac{\sqrt{2\pi}}{2} \operatorname{errfn}'\left(\frac{\tau}{2}\right) \end{aligned}$$

so

$$\frac{1}{\sqrt{2\pi}} g'(\tau) = \frac{1}{2} \operatorname{errfn}'\left(\frac{\tau}{2}\right)$$

integrate this equation to get

$$\frac{1}{\sqrt{2\pi}} g(\tau) = \operatorname{errfn}\left(\frac{\tau}{2}\right) - \frac{1}{2}$$

because $g(0) = 0$ and $\operatorname{errfn}(0) = \frac{1}{2}$.

Remark 8.5. The reason that we made the excited term $E(x,t;y)$ look like (8.4) is that we would like to have the Green's function decompositon look similar to the scalar Burger's equation case in [Z1], doing so also makes $E(x,t;y)$ vanishes at $t = 0$.

Remark 8.6. The $\frac{e^{-x}}{e^x + e^{-x}}$ term in the scattering term serves as a smooth cutoff function, which smoothly interpolate between different cases of solutions. For $x > 0$ and $|x|$ large, $\frac{e^{-x}}{e^x + e^{-x}}$ decays to 0, for $x < 0$ and $|x|$ large, $\frac{e^{-x}}{e^x + e^{-x}}$ is almost 1.

Now it is time to give some L^p estimates on Green's function convolved with some function f in $L^p(\mathbb{R})$ $(1 \le p \le \infty)$.

Proposition 8.7. *The Green's function G decomposes as $G = E + S + R = E + \tilde{G}$, where $\tilde{G} = S + R$, $E(x,t;y) = C\bar{u}'(x)e(y,t)$ and*

$$e(y,t) := \operatorname{errfn}\left(\frac{y + f'(\bar{u}_-)t}{\sqrt{4t}}\right) - \operatorname{errfn}\left(\frac{y - f'(\bar{u}_-)t}{\sqrt{4t}}\right)$$

then for some $C > 0$ and all $t > 0$,

$$\left| \int_{-\infty}^{+\infty} \tilde{G}(x,t;y) f(y) dy \right|_{L^p(x)} \le C t^{-\frac{1}{2}(1-\frac{1}{p})} |f|_{L^1}, \tag{8.41}$$

$$\left|\int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y)f(y)dy\right|_{L^p(x)} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}|f|_{L^1}, \tag{8.42}$$

$$\left|\int_{-\infty}^{+\infty} \tilde{G}(x,t;y)f(y)dy\right|_{L^p(x)} \le C|f|_{L^p}, \tag{8.43}$$

$$\left|\int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y)f(y)dy\right|_{L^p(x)} \le Ct^{-\frac{1}{2}}|f|_{L^p}. \tag{8.44}$$

and

$$\left|\int_{-\infty}^{+\infty} e(y,t)f(y)dy\right| \le C|f|_{L^1}, \left|\int_{-\infty}^{+\infty} e_y(y,t)f(y)dy\right| \le Ct^{-\frac{1}{2}}|f|_{L^1}, \tag{8.45}$$

$$\left|\int_{-\infty}^{+\infty} e_t(y,t)f(y)dy\right| \le Ct^{-\frac{1}{2}}|f|_{L^1}, \left|\int_{-\infty}^{+\infty} e_{ty}(y,t)f(y)dy\right| \le Ct^{-1}|f|_{L^1}, \tag{8.46}$$

$$\left|\int_{-\infty}^{+\infty} e_t(y,t)f(y)dy\right| \le C|f|_{L^\infty}, \left|\int_{-\infty}^{+\infty} e_{yt}(y,t)f(y)dy\right| \le Ct^{-\frac{1}{2}}|f|_{L^\infty}. \tag{8.47}$$

Proof. We prove (8.41) first. Write $\tilde{G}$ as $\tilde{G}(x,t;y) = S(x,t;y) + R(x,t;y)$, recall from Proposition 8.1 that

$$S(x,t;y) = \chi_{\{t\ge 1\}}(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \left(\frac{e^{-x}}{e^x+e^{-x}}\right)$$

so

$$|S(x,t;y)| \le (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \tag{8.48}$$

and

$$R(x,t;y) = \mathbf{O}\left(e^{-\eta t}e^{-\frac{|x-y|^2}{Mt}}\right) + \mathbf{O}\left((t+1)^{-\frac{1}{2}}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}$$

so

$$|R(x,t;y)| \le Ce^{-\eta|x|}t^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}} + Ce^{-\eta'(|x-y|+t)} \tag{8.49}$$

for some $0 < \eta' < \eta$. By Minkowski's inequality,

$$\left|\tilde{G}(x,t;y)\right|_{L^p(x)} \le |S(x,t;y)|_{L^p(x)} + |R(x,t;y)|_{L^p(x)}$$

We estimate $|S(x,t;y)|_{L^p(x)}$ first,

$$\begin{aligned}
|S(x,t;y)|_{L^p(x)} &\leq \left(\int_{-\infty}^{+\infty}\left[(4\pi t)^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}}\right]^p dx\right)^{\frac{1}{p}}\\
&\leq (4\pi t)^{-\frac{1}{2}}\left(\int_{-\infty}^{+\infty}e^{-\frac{p}{4t}(x-y-f'(\bar{u}_-)t)^2}dx\right)^{\frac{1}{p}}\\
&= (4\pi t)^{-\frac{1}{2}}\left(\sqrt{\frac{4t}{p}}\int_{-\infty}^{+\infty}e^{-z^2}dz\right)^{\frac{1}{p}}\\
&= (4\pi t)^{-\frac{1}{2}}\left(\sqrt{\frac{4t}{p}}\sqrt{\pi}\right)^{\frac{1}{p}}\\
&= C_1 t^{-\frac{1}{2}(1-\frac{1}{p})}
\end{aligned}$$

Then we estimate $|R(x,t;y)|_{L^p(x)}$,

$$\begin{aligned}
|R(x,t;y)|_{L^p(x)} &\leq \left|Ce^{-\eta|x|}t^{-\frac{1}{2}}e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}\right|_{L^p(x)} + \left|Ce^{-\eta'(|x-y|+t)}\right|_{L^p(x)}\\
&:= R_A + R_B
\end{aligned}$$

Notice that R_A is similar to $|S(x,t;y)|_{L^p(x)}$, so we have

$$R_A \leq C_2 t^{-\frac{1}{2}(1-\frac{1}{p})}$$

Now we estimate R_B,

$$\begin{aligned}
R_B^p &= C^p\int_{-\infty}^{+\infty}\left(e^{-\eta'(|x-y|+t)}\right)^p dx\\
&= C^p\int_{-\infty}^{+\infty}e^{-p\eta'(|x-y|+t)}dx\\
&= C^p\int_{-\infty}^{y}e^{-p\eta'(y-x+t)}dx + C^p\int_{y}^{+\infty}e^{-p\eta'(x-y+t)}dx\\
&= C^p e^{-p\eta' t}\left(\int_{-\infty}^{y}e^{-p\eta'(y-x)}dx + \int_{y}^{+\infty}e^{-p\eta'(x-y)}dx\right)\\
&= 2C^p e^{-p\eta' t}\int_{0}^{+\infty}e^{-p\eta' x}dx\\
&= \frac{2C^p e^{-p\eta' t}}{p\eta'}
\end{aligned}$$

so

$$R_B = C\left(\frac{2}{p\eta'}\right)^{\frac{1}{p}}e^{-\eta' t}$$

Finally, we use the above estimates to derive,

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} \tilde{G}(x,t;y)f(y)dy\right|_{L^p(x)}\\
\le\ & \int_{-\infty}^{+\infty} \left|\tilde{G}(x,t;y)\right|_{L^p(x)} |f(y)|\, dy\\
\le\ & \int_{-\infty}^{+\infty} \left(|S(x,t;y)|_{L^p(x)} + |R(x,t;y)|_{L^p(x)}\right) |f(y)|\, dy\\
\le\ & \int_{-\infty}^{+\infty} \left(C_1 t^{-\frac{1}{2}(1-\frac{1}{p})} + C_2 t^{-\frac{1}{2}(1-\frac{1}{p})} + C\left(\frac{2}{p\eta'}\right)^{\frac{1}{p}} e^{-\eta' t}\right) |f(y)|\, dy\\
\le\ & \int_{-\infty}^{+\infty} (C_1+C_2+C_3) t^{-\frac{1}{2}(1-\frac{1}{p})} |f(y)|\, dy\\
=\ & C t^{-\frac{1}{2}(1-\frac{1}{p})} |f|_{L^1}.
\end{aligned}$$

For y-derivative bounds of $\tilde{G}$, $\tilde{G}_y(x,t;y)$, we just need to notice that we have the following estimates

$$|S_y(x,t;y)| \le C t^{-1} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \tag{8.50}$$

and

$$R_y(x,t;y) = \mathbf{O}\left(e^{-\eta t} e^{-\frac{|x-y|^2}{Mt}}\right) + \mathbf{O}\left((t+1)^{-\frac{1}{2}} e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}$$

so

$$|R_y(x,t;y)| \le C e^{-\eta|x|} t^{-1} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}} + C e^{-\eta'(|x-y|+t)} \tag{8.51}$$

for some $0 < \eta' < \eta$. With some similar computation as the for $\tilde{G}(x,t;y)$ we get

$$\left|\int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y)f(y)dy\right|_{L^p(x)} \le C t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} |f|_{L^1},$$

To prove (8.43) and (8.44), notice that (8.48) implies

$$|S(x,t;y)|_{L^1(x)} \le \left|(4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}}\right|_{L^1(x)} \le C \tag{8.52}$$

(8.49) implies

$$|R(x,t;y)|_{L^1(x)} \le C\left|e^{-\eta|x|} t^{-\frac{1}{2}} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}}\right|_{L^1(x)} + C\left|e^{-\eta'(|x-y|+t)}\right|_{L^1(x)} \le C \tag{8.53}$$

(8.50) implies

$$|S_y(x,t;y)|_{L^1(x)} \leq C \left| t^{-1} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{4t}} \right|_{L^1(x)} \leq Ct^{-\frac{1}{2}} \tag{8.54}$$

(8.51) implies

$$|R_y(x,t;y)|_{L^1(x)} \leq C \left| e^{-\eta|x|} t^{-1} e^{-\frac{(x-y-f'(\bar{u}_-)t)^2}{Mt}} \right|_{L^1(x)} + C \left| e^{-\eta'(|x-y|+t)} \right|_{L^1(x)} \leq Ct^{-\frac{1}{2}} \tag{8.55}$$

Use estimates (8.52) – (8.55) and the inequality $|f * g|_{L^p} \leq |f|_{L^1}|g|_{L^p}$ we can derive the estimates (8.43) and (8.44). □

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