Stability of viscous shocks in isentropic gas dynamics

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Overview

The p-system

The isentropic compressible Navier-Stokes equations in one spatial dimension, or p-system with real viscosity:

$$v_t - u_x = 0,$$
$$u_t + p(v)_x = \left(\frac{u_x}{v}\right)_x,$$

where, physically, v is the specific volume, u is the velocity, and $p(v)$ is the pressure law (γ law-adiabatic).

What is a viscous shock profile?

By a viscous shock profile we mean a traveling wave solution

$$v(x,t) = \hat{v}(x - st),$$
$$u(x,t) = \hat{u}(x - st),$$

moving with speed s and having asymptotically constant end-states $(v_\pm, u_\pm)$.

Moving Frame

Translate $x \to x - st$

$$v_t - sv_x - u_x = 0,$$
$$u_t - su_x + (a_0 v^{-\gamma})_x = \left(\frac{u_x}{v}\right)_x.$$

Re-scale $(x, t, v, u) \to (-\varepsilon s x, \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s))$, where ε is chosen so that $0 < v_+ < v_- = 1$. Our system takes the form

$$v_t + v_x - u_x = 0,$$
$$u_t + u_x + (a v^{-\gamma})_x = \left(\frac{u_x}{v}\right)_x.$$

Profile ODE

The shock profile satisfies the ordinary differential equation

$$v' - u' = 0,$$
$$u' + (av^{-\gamma})' = \left(\frac{u'}{v}\right)',$$

subject to $(v(\pm\infty), u(\pm\infty)) = (v_\pm, u_\pm)$. This simplifies to

$$v' + (av^{-\gamma})' = \left(\frac{v'}{v}\right)'.$$

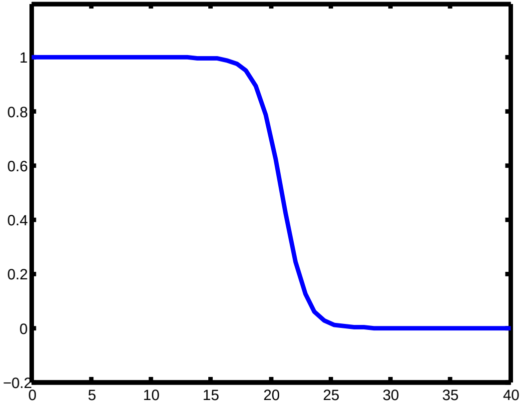
By integrating from $-\infty$ to x, we get the profile equation

$$v' = v(v - 1 + a(v^{-\gamma} - 1)),$$

Solution

Remark

Since the profile is first order scalar, it has a monotone solution. Since $v_+ < v_-$, we have that $\hat{v}_x < 0$ for all $x \in \mathbb{R}$.

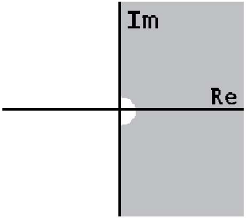


What do we mean by stability?

Definition

We say that a shock profile is *spectrally stable* if the linearized system has no spectrum in the closed deleted right half-plane

$$P = \{\Re e(\lambda) \geq 0\} \setminus \{0\}.$$



Note

Zumbrun & Howard [1998, 2000], and Mascia & Zumbrun [2003,2004] proved that spectral stability implies asymptotic orbital stability, that is, a perturbed wave converges to a translate of the original profile.

The Eigenvalue Problem

Linearize about the profile $(\hat{v}, \hat{u})$:

$$\lambda v + v' - u' = 0,$$
$$\lambda u + u' - \left(\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v\right)' = \left(\frac{u'}{\hat{v}}\right)',$$

where

$$h(\hat{v}) = -\hat{v}^{\gamma+1} + a(\gamma - 1) + (a+1)\hat{v}^{\gamma}$$

Integrated Coordinates

- Coordinate change, $(u, v) \to (u', v')$, then integrate. Reduces to integrated eigenvalue problem.

$$\lambda v + v' - u' = 0,$$
$$\lambda u + u' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v' = \frac{u''}{\hat{v}}.$$

- Removes the eigenvalue at the origin - important for uniform bounds in energy estimates.

Small amplitude spectral stability

Theorem (Matsumura and Nishihara)

Viscous shocks of our system are spectrally stable whenever

$$\left(\frac{v_+^{\gamma+1}}{a\gamma}\right)^2 + 2(\gamma-1)\left(\frac{v_+^{\gamma+1}}{a\gamma}\right) - (\gamma-1) \geq 0.$$

In particular, as $v_+ \to 1$ (hence $a\gamma \to 1$), the left-hand side approaches γ and so the inequality is satisfied. Therefore, small-amplitude viscous shocks are spectrally stable.

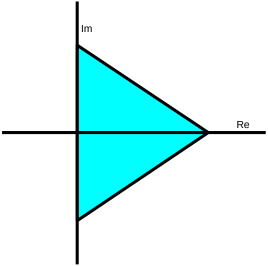
The proof follows from a clever energy estimate.

Our Theorem: High frequency bounds

Theorem (B. H. R. Z.)

Unstable spectrum cannot exist outside of the triangular region given by

$$(\Re e(\lambda) + |\Im m(\lambda)|) \leq (\sqrt{\gamma} + \frac{1}{2})^2.$$



Lemma 1

The following identity holds for $\Re e \lambda \geq 0$:

$$(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}} \hat{v}|u|^2 - \frac{1}{2} \int_{\mathbb{R}} \hat{v}_x |u|^2 + \int_{\mathbb{R}} |u'|^2$$
$$\leq \sqrt{2} \int_{\mathbb{R}} \frac{h(\hat{v})}{\hat{v}^\gamma} |v'||u| + \int_{\mathbb{R}} \hat{v}|u'||u|$$

Proof

We multiply the second equation in the eigenvalue problem by $\hat{v}\bar{u}$ and integrate along x. This yields

$$\lambda \int_{\mathbb{R}} \hat{v}|u|^2 + \int_{\mathbb{R}} \hat{v}u'\bar{u} + \int_{\mathbb{R}} |u'|^2 = \int_{\mathbb{R}} \frac{h(\hat{v})}{\hat{v}^\gamma} v'\bar{u}.$$

By taking the real and imaginary parts and adding them together, and noting that $|\Re e(z)| + |\Im m(z)| \leq \sqrt{2}|z|$ we obtain the inequality. □

Lemma 2

The following identity holds for $\Re e\lambda \geq 0$:

$$\int_{\mathbb{R}} |u'|^2 = 2\Re e(\lambda)^2 \int_{\mathbb{R}} |v|^2 + \Re e(\lambda) \int_{\mathbb{R}} \frac{|v'|^2}{\hat{v}} + \frac{1}{2} \int_{\mathbb{R}} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}} \right] |v'|^2$$

Proof

Involves standard energy estimate techniques.

Proof of Theorem

Young's inequality applied twice with the right-hand side of inequality in Lemma 1, then with Lemma 2 we get

$$(\Re e(\lambda) + |\Im m(\lambda)|) < \frac{(4\gamma - 1)\epsilon - 1}{4\epsilon(1-\epsilon)}.$$

Setting $\epsilon = 1/(2\sqrt{\gamma} + 1)$ gives the bound

$$(\Re e(\lambda) + |\Im m(\lambda)|) \leq (\sqrt{\gamma} + \frac{1}{2})^2.$$

Eigenvalue Problem

Write the eigenvalue problem

$$\lambda v = Lv, \qquad -\infty < x < \infty,$$

as a first order system

$$\begin{cases} W' = A(x,\lambda)W, & W \in \mathbb{C}^n \\ W(\pm\infty) = 0. \end{cases}$$

$$A(x,\lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\hat{v} & \lambda\hat{v} & f(\hat{v}) - \lambda \end{pmatrix}, \quad W = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}, \quad ' = \frac{d}{dx},$$

and $f(\hat{v}) = \hat{v} - \hat{v}^{-\gamma} h(\hat{v})$

Evans Function Computation

Definition

Define

$$D(\lambda) = \underbrace{W_1^- \wedge \ldots \wedge W_k^-}_{U^-(\lambda)} \wedge \underbrace{W_{k+1}^+ \wedge \ldots \wedge W_n^+}_{S^+(\lambda)}$$

where $\{W_i^-\}_{i=1}^k$ and $\{W_j^+\}_{j=k+1}^n$ are analytic bases of the unstable/stable manifolds at $x = \mp\infty$, respectively.

Evans Function Computation

- 1-D Unstable manifold $W_1^-(x)$ at $x = -\infty$
- 2-D Stable manifold $W_2^+(x) \wedge W_3^+(x)$ at $x = \infty$
- $D(\lambda) := \det(W_1^- W_2^+ W_3^+)_{|x=0}$ is zero when λ is an eigenvalue.

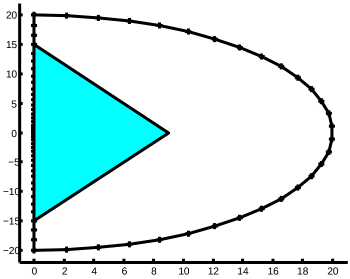
Alternative Adjoint Formulation

- $\widetilde{W}' = -\widetilde{W} A(x, \lambda)$
- Note that $\widetilde{W}_1^+(x) \perp W_2^+(x) \wedge W_3^+(x)$
- $D(\lambda) = 0 \Leftrightarrow D_+(\lambda) := (\widetilde{W}_1^+ W_1^-)_{|x=0} = 0$.

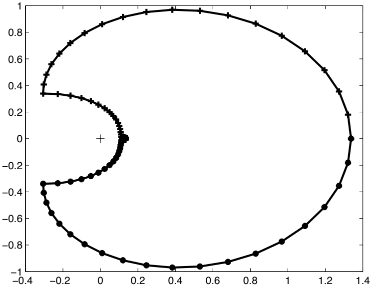
Improved Numerics

- Let $W(x) = e^{\mu^{\mp}x}V(x)$, where $\mu^{\mp}$ is the growth rate of the unstable manifold at $x = \mp\infty$.
- Initialize $V(x)$ at $x = \mp\infty$ to be the eigenvector of $A(\mp\infty, \lambda)$ corresponding to $\mu^{\mp}$.
- Scale out exponential growth/decay, $V' = (A(x,\lambda) - \mu^{\pm})V(x)$. Similarly for $\widetilde{V}(x)$.

Evans Function



(a)



(b)

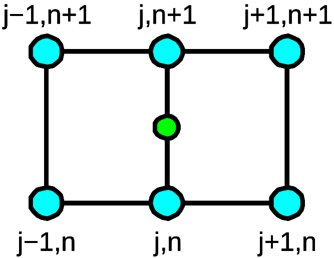
$\gamma = 5/3$, $v_+ = 1 \times 10^{-4}$, $M \approx 1669$.

Finite difference simulation

Implicit/Explicit Scheme similar to Crank-Nicholson, with a Newton solver for non-linearity:

$$v_t + v_x - u_x = 0,$$
$$u_t + u_x - a\gamma v^{-\gamma-1} v_x = \frac{u_{xx}}{v} - \frac{u_x v_x}{v^2}.$$

Let n and j be, respectively, the discretized temporal and spacial indices.



Difference Equation

$$\frac{v_j^{n+1}-v_j^n}{\Delta t}+\frac{1}{4\Delta x}(v_{j+1}^{n+1}-v_{j-1}^{n+1}+v_{j+1}^n-v_{j-1}^n)$$
$$-\frac{1}{4\Delta x}(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^n-u_{j-1}^n)=0,$$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{1}{4\Delta x}(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^n-u_{j-1}^n)$$
$$-\frac{a}{4\Delta x}\gamma(v_j^n)^{-\gamma-1}(v_{j+1}^{n+1}-v_{j-1}^{n+1}+v_{j+1}^n-v_{j-1}^n)$$
$$-\frac{1}{2(\Delta x)^2 v_j^n}(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}+u_{j+1}^n-2u_j^n+u_{j-1}^n)$$
$$+\frac{1}{16(\Delta x)^2(v_j^n)^2}(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^n-u_{j-1}^n)$$
$$\times(v_{j+1}^{n+1}-v_{j-1}^{n+1}+v_{j+1}^n-v_{j-1}^n)=0.$$

Isentropic Gas with Capillarity

Slemrod's model/Isentropic Navier-Stokes with capillarity:

$$v_t - u_x = 0$$

$$u_t + p(v)_x = \left(\frac{u_x}{v}\right)_x + dv_{xxx}.$$

Physically, $d < 0$ is the dispersion coefficient.

Numerics

$$F = \frac{1}{\Delta t}(v_j^{n+1} - v_j^n) + \frac{1}{24\Delta x}(v_{j-2}^{n+1} - 8v_{j-1}^{n+1} + 8v_{j+1}^{n+1} - v_{j+2}^{n+1} + v_{j-2}^n - 8v_{j-1}^n + 8v_{j+1}^n - v_{j+2}^n)$$

$$- \frac{1}{24\Delta x}(u_{j-2}^{n+1} - 8u_{j-1}^{n+1} + 8u_{j+1}^{n+1} - u_{j+2}^{n+1} + u_{j-2}^n - 8u_{j-1}^n + 8u_{j+1}^n - u_{j+2}^n) = 0$$

$$G = \frac{1}{\Delta t}(u_j^{n+1} - u_j^n) + \frac{1}{24\Delta x}(u_{j-2}^{n+1} - 8u_{j-1}^{n+1} + 8u_{j+1}^{n+1} - u_{j+2}^{n+1} + u_{j-2}^n - 8u_{j-1}^n + 8u_{j+1}^n - u_{j+2}^n)$$

$$- \frac{a\gamma(v_j^n)^{-\gamma-1}}{24\Delta x}(v_{j-2}^{n+1} - 8v_{j-1}^{n+1} + 8v_{j+1}^{n+1} - v_{j+2}^{n+1} + v_{j-2}^n - 8v_{j-1}^n + 8v_{j+1}^n - v_{j+2}^n)$$

$$- \frac{1}{24(v_j^n)(\Delta x)^2}(-u_{j-2}^{n+1} + 16u_{j-1}^{n+1} - 30u_j^{n+1} + 16u_{j+1}^{n+1} - u_{j+2}^{n+1} - u_{j-2}^n + 16u_{j-1}^n - 30u_j^n + 16u_{j+1}^n - u_{j+2}^n)$$

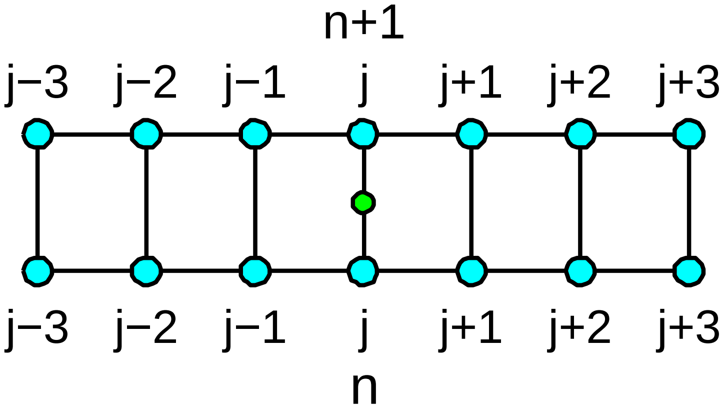
$$+ \frac{1}{24(v_j^n)^2(\Delta x)^2}(v_{j-2}^{n+1} - 8v_{j-1}^{n+1} + 8v_{j+1}^{n+1} - v_{j+2}^{n+1} + v_{j-2}^n - 8v_{j-1}^n + 8v_{j+1}^n - v_{j+2}^n)$$

$$\times(u_{j-2}^{n+1} - 8u_{j-1}^{n+1} + 8u_{j+1}^{n+1} - u_{j+2}^{n+1} + u_{j-2}^n - 8u_{j-1}^n + 8u_{j+1}^n - u_{j+2}^n)$$

$$- \frac{d}{16(\Delta x)^3}(v_{j-3}^{n+1} - 8v_{j-2}^{n+1} + 13v_{j-1}^{n+1} - 13v_{j+1}^{n+1} + 8v_{j+2}^{n+1} - v_{j+3}^{n+1}$$

$$+v_{j-3}^n - 8v_{j-2}^n + 13v_{j-1}^n - 13v_{j+1}^n + 8v_{j+2}^n - v_{j+3}^n) = 0$$

Numerics



p-system

Jeffrey Humpherys, Olivier Lafitte, and Kevin Zumbrun. "Stability of isentropic viscous shock profiles in the high mach number limit." Communications in Mathematical Physics, 293(1):1-36, 2010.