Stability of viscous shock waves and beyond

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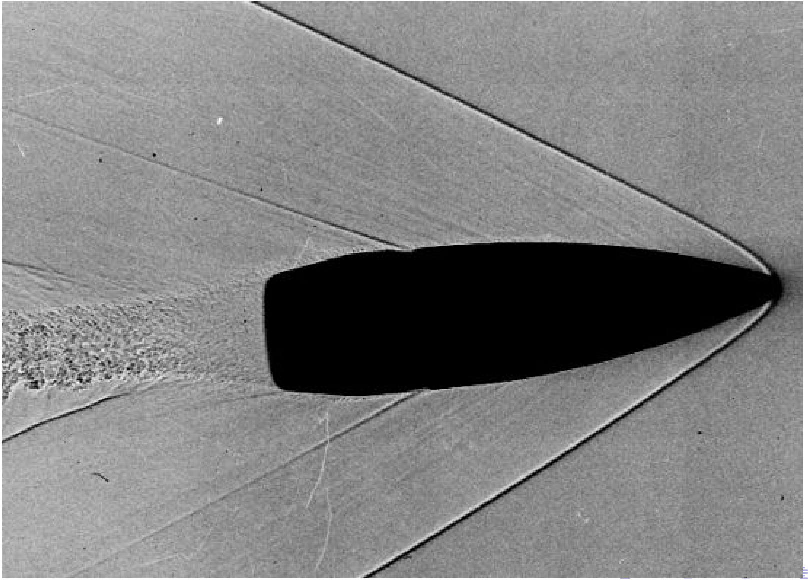
Bow shock in compressible gas



Quasilinear hyperbolic–parabolic conservation laws:

$$u_t + f(u)_x = \begin{cases} 0 \\ \nu(b(u)u_x)_x \end{cases}, \quad u \in R^n,\ \nu > 0,$$

govern compressible (gas, plasma, solid) mechanics. Interesting solutions [Riemann1856] [Rayleigh1919,Gilbarg1951] are shock waves

$$u(x,t) = \bar{u}\Big(\frac{x - st}{\nu}\Big), \qquad \lim_{x \to \pm\infty} u(x) = u_\pm,$$

propagating large energies coherently over great distance,
Stationary solutions of $u_t - su_x + f(u)_x = \nu(b(u)u_x)_x$.

Entropy and symmetrizability

Typically associated to $u_t + f(u)_x = \nu(bu_x)_x$ is an **entropy**

$$\eta(u), \qquad d\eta df = dq, \qquad d^2\eta(u)b \geq 0, \qquad \partial_t \int \eta(u) \leq 0.$$

Convexity of $\eta \Rightarrow$ symmetric hyperbolicity of $u_t + f(u)_x = 0$ (in strong, *nonlinear* sense that $u(w)_t + f(u(w))_x = 0$ symmetric), *Thermodynamic stability* (stability of homogeneous state).

We'll assume hyperbolicity, usually verified by entropy considerations, *only at endstates* $u_\pm$, hence profiles can cross elliptic regions ($\sim$ phase-transition, ionization, etc.).

(For the examples considered, hyperbolicity $\Leftrightarrow$ symmetric hyperbolicity $\Leftrightarrow$ existence of a convex entropy.)

Example systems

1D isentropic gas dynamics, $\tau, u \in R$ [Weyl,Gilbarg]:

$$\tau_t - u_x = 0,$$
$$u_t + p(\tau)_x = \left(\frac{u_x}{\tau_d}\right),$$

$\tau =$ specific volume, $u =$ velocity, $p =$ pressure.

Entropy (mechanical energy) $\eta = \int_\tau^{+\infty} p(z)dz + \frac{|u|^2}{2}$; hyperbolic wherever p monotone decreasing $\Leftrightarrow \eta$ convex.

(Lagrangian coordinates; equivalently, nonlinear wave equation, $u = \xi_t$, $\tau = \xi_x$.)

Example systems

2D isentropic MHD, $\tau \in R$, $u, B \in R^2$ [Germain,Conley–Smoller]:

$$\tau_t - u_{1,x} = 0,$$
$$(\tau B_2)_t - (B_1^* u_2)_x = 0,$$
$$u_{1t} + \left(p(\tau) + \left(\frac{B_2^2}{2\mu_0}\right)\right)_x = \left(\frac{(2\mu+\eta)u_{1x}}{\tau}\right)_x,$$
$$u_{2t} - \left(\frac{B_1^* B_2}{\mu_0}\right)_x = \left(\frac{\mu u_{2x}}{\tau}\right)_x,$$

$B_1^* \equiv \text{constant}$ ($\Leftrightarrow \nabla \cdot B = 0$), $\mu, \eta > 0$, $p(\tau) = c\tau^{-\gamma}$, $\gamma > 1$.

Entropy $\eta = \int_\tau^{+\infty} p(z)dz + \frac{|u|^2}{2} + \frac{\tau |B|^2}{2\mu_0}$; hyperbolic wherever p monotone $\Leftrightarrow$ η convex (everywhere).

Example systems

2D viscoelasticity $\tau, u \in R^2$ [Antman–Malek-Madani,BLeZ]:

$$\tau_t - u_x = 0,$$
$$u_t + dW(\tau)_x = \left(\frac{u_x}{\tau_1}\right),$$

$\tau = \xi_x =$ strain, $u = \xi_t =$ velocity, $W = \frac{1}{2}(1+\tau_2^2) + \frac{1}{4}(|\tau|^2 - 1)^2$ elastic potential. (Inherited from

$$W(\nabla_{x,y,z}\xi) = \frac{1}{4}|\nabla_{x,y,z}\xi^T \nabla_{x,y,z}\xi - \mathrm{Id}|^2,$$

measuring squared distance from $SO(2)$.)

Entropy (mechanical energy) $\eta = W + \frac{|u|^2}{2}$; hyperbolic wherever W convex (compare full 2D case, rank one convexity*).

Stability Theory

Studied intensively since 1960's (1980's) [Erpenbeck, Landau, Dy'akov,...] [Oleinik, Matsumura, Nishihara, Kawashima, Goodman, Liu, ...] In principle, well understood.

Stability criteria: reduce to spectra of linearized equations, computable as zeros of a *Lopatinski* (*Evans*) determinant [Majda1983, Métivier1992,...][Gardner,Howard, Mascia, Serre, Zumbrun, 1998-2006].

All-parameters stability analyses: Combining asympototic analysis/singular perturbations and numerical Evans analysis, determine stability across full parameter range [Barker, Humpherys, Lafitte, Lewicka, Lyng, Zumbrun, 2009-2012].

(Abbreviated) Viscous stability literature

Until recently (cf. $\exists$, 1951), stability mostly open for $\nu > 0$.

Scalar equations.
General result: [Sattinger1976], *weighted norms*.

Small amplitude Lax shocks.
Zero-mass perturbations: [Goodman1986] and [Matsumura-Nishihara85,Kawashima-M-N1986].
Identity viscosity: [Szepessy-Xin1993] (see also [Liu1986]).*
Physical viscosity: [Humpherys-Z2002] and [Mascia-Z2004].*

Large amplitude/nonclassical shocks.
Instability criteria: [Gardner-Z1998], [Z-Serre1999], ...
Stability criteria: [Z-Howard1998], [Mascia-Z2004], ...

VERIFICATION: (main current direction.)

General program (largely complete)

Describe existence and stability *in the large*, i.e., across the full physical parameter range (but finite basin of attraction).

- Profile equations and gradient flow.
- Abstract stability framework: spectral vs. nonlinear stability.
- Numerical and analytical verification; large-parameter limits.

Remarks.
1. Stability framework of much wider application: conditional stability and bifurcation, stability of time-periodic shocks, multi-dimensional stability, stability of relaxation shocks and detonations, modulational stability of periodic wave trains*.
2. Numerical Evans function methods are system independent.
3. Large-parameter limits useful as "organizing centers" for intermediate parameters, regardless of physical validity.

II. Traveling wave profiles

Substituting $u = \bar{u}(x - st)$ into $u_t + f(u)_x = \nu(b(u)u_x)_x$ and integrating in x yields ODE (dimension rank b=1,2)

$$\nu b(\bar{u})\bar{u}' = f(\bar{u}) - s\bar{u} - q.$$

When there exists an entropy η, this can be written as

$$\nu d^2\eta b(u)u' = \nabla_u \phi(u),$$

$$\phi(u) := s\eta - q + d\eta(f(U) - f(U_-) - s(u - u_-)),$$

for η convex a **generalized gradient flow** increasing ϕ.

Type of shock (Lax, over/undercompressive) $\sim$ relative Morse index of endpoints [Mascia-Z2003,BLZ2009].

Gas dynamics and viscoelasticity: reduced equivalent flow

Writing the profile equation as

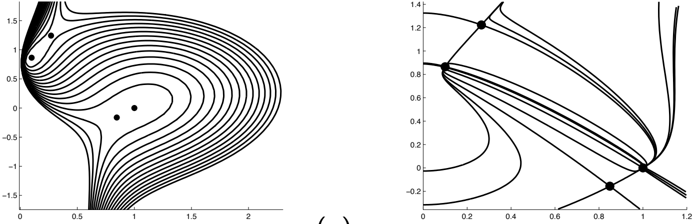
$$-s\tau' - u' = 0,$$
$$-su' - dW(\tau)' = \left(\frac{u'}{\tau_1}\right)',$$

substituting the first equation into the second, making the change of coordinates $z \to sz$, and defining $\sigma = s^2$, we obtain the *reduced profile equation* $-\sigma\tau' + dW(\tau)' = \left(\frac{\tau'}{\tau_1}\right)'$ associated with

$$\tau_t + dW(\tau)_z = \left(\frac{\tau_z}{\tau_1}\right)_z,$$

convex entropy $\tilde{\eta}(\tau) = \frac{|\tau|^2}{2}$, $\Rightarrow$ gradient flow **even in elliptic regions** (hence no periodic or homoclinic orbits).

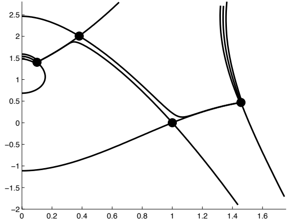
Phase portraits: MHD



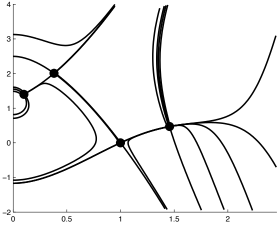
(a) (b)

Figure: Typical phase portrait for MHD with two variables and infinite electric resistivity ($\sigma = \infty$). Parameter values are $\gamma = 2$, $v_+ = 0.1$, $I = 0.7$, $B_{2+} = 0.7$, and $\mu_0 = 1$. In Figure (a) we plot level sets of ϕ and in Figure (b) we draw the phase portrait.

Phase portraits: MHD



(*a*)



(*b*)

Figure: Transition to undercompressive profile. keeping $\tau = 2\mu + \eta = 1$ and letting $\mu \to 0$, we find that the overcompressive family is squeezed to an undercompressive connection somewhere between $\mu = 0.185$ (Figure (a)) and $\mu = 0.17$ (Figure (b)).

Phase portraits: viscoelasticity

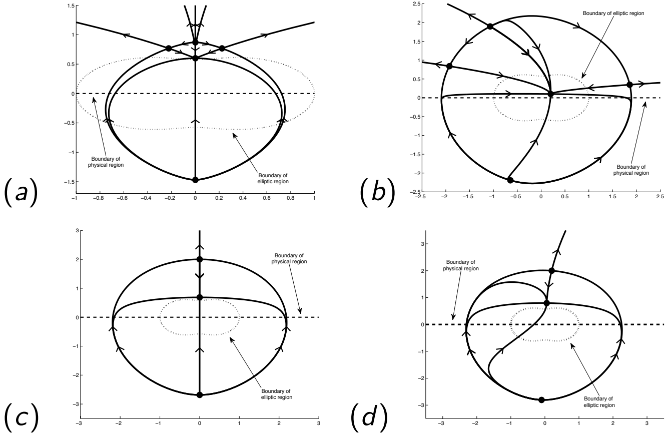


Figure: Typical three and five-equilibrium phase portraits for the 2-D compressible case. The dark dashed lines bound the physically relevant region $\tau_1 > 0$ and the light dashed lines surround the elliptic region.

Conclusions

Rich solution structure, many types, configurations of shocks.

FAR FROM PARADIGM OF WEAK ($\sim$ scalar) LAX SHOCK: no simple description even of background profile...

(Proofs by vector index, planar phase diagram analysis.)

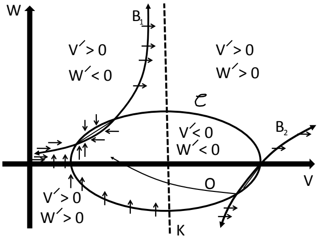


Figure: Sample phase diagram analysis.

III. Stability framework: spectral vs. nonlinear stability

Take WLOG $s = 0$ (co-moving coordinates), $\nu = 1$, so that $u = \bar{u}(x)$ is an equilibrium. Linearized eigenvalue equations

$$\lambda u = Lu := -(A(x)u)_x + (B(x)u_x)_x$$

may be written as a first-order system

$$W' = A(x, \lambda)W, \tag{1}$$

where W is an augmented "phase variable" including u and suitable derivatives.

Define the *Evans function*

$$D(\lambda) := \det(W_1^-, \ldots, W_k^-, W_{k+1}^+, \ldots, W_N)|_{x=0} \tag{2}$$

in terms of analytically-chosen bases $\{W_1^-, \ldots, W_k^-\}(\lambda, x)$ and $\{W_{k+1}^+, \ldots, W_N)\}(\lambda, x)$ of the manifolds of solutions decaying as $x \to \infty$ and $x \to +\infty$ of (1) [Mascia-Z2003].

Properties

Lemma (Gardner-Z1998,Mascia-Z2003)

D may be defined analytically on $\Re\lambda \geq 0$. Zeros for $\lambda \neq 0$ correspond to eigenvalues of L. D vanishes to at least order ℓ at $\lambda = 0$, where ℓ is the dimension of the manifold of nearby shock connections (at least one, by translation invariance).

Lemma (Gardner-Z1998,Z-Serre1999,Mascia-Z2003)

$\partial_\lambda^\ell D(0) \neq 0 \Leftrightarrow$ *transversality of the profile connection plus hyperbolic stability of the corresponding inviscid shock ($\nu = 0$).*

Stability criterion

(H) Symmetrizability at $u_\pm$ plus nondegeneracy hypotheses.
(D) D has ℓ zeros on $\Re\lambda \geq 0$, all at $\lambda = 0$.

Theorem (Mascia-Z2004,Howard-Z2006,Raoofi-Z2009)

Let $\tilde{u}$ be a solution of $u_t + f(u)_x = \nu(b(u)u_x)_x$. Assuming (H), (D), and $E_0 := \|(1+|x|^2)^{\frac{3}{4}}(\tilde{u}-\bar{u})\|_{H^4}|_{t=0}$ sufficiently small,

$$\|\tilde{u} - \bar{u}(\cdot - \psi)\|_{L^p}(t) \leq C(1+t)^{-\frac{1}{2}(1-1/p)}E_0,$$
$$|\psi_t| \leq C(1+t)^{-1}E_0,$$
$$|\psi - \psi(+\infty)| \leq CE_0(1+t)^{-\frac{1}{2}}$$

for some $C > 0$, $\psi(t)$ and all $t \geq 0$, $1 \leq p \leq \infty$.

Ingredients of proof: Green function bounds

Lemma

Under (H), (D), $G = E + \tilde{G}$, where $E = \bar{u}'(x)e(t;y)$, $|e|_{L^\infty} \leq C$, and $\tilde{G}$, e_t decay like summations of Gaussians.

Proof.

Inverse Laplace transform representation

$$G(x,t;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)d\lambda,$$

($G_\lambda = (\lambda - L)^{-1}\delta_y(x) =$ resolvent kernel), plus asymptotic ODE and stationary phase estimates. Pole terms $G_\lambda \sim \sum_j \frac{m_j(\lambda)\bar{u}_x\tilde{\phi}_j(y)}{D(\lambda)}$ lead to contribution $\bar{u}_x e$, role of condition (D). □

Nonlinear cancellation estimate

Given a solution $\tilde{u}(x,t)$, define $v = u - \bar{u} = \tilde{u}(x + \psi(x,t)) - \bar{u}(x)$.

Lemma

$(\partial_t - L)\, v = (\partial_t - L)\, \bar{u}'(x_1)\psi + Q_x$, *where* $Q = \mathcal{O}(|(v, v_x, \psi_t)|^2)$.

Contribution of source $(\partial_t - L)\, \bar{u}'(x_1)\psi$ is evidently $\bar{u}'(x_1)\psi$. Defining

$$\psi(t) = -\int_0^t \int e(t-s; y)(R_x dy\, ds$$

cancels "bad" term $\bar{u}'(x)e$ in $G = \bar{u}'e + \tilde{G}$.

Nonlinear iteration

Resulting closed system in (v, ψ_t):

$$v(x,t) = \int \tilde{G}(x,t;y)v_0 dy\, ds + \int_0^t \int \tilde{G}(x,t-s;y)Q_x dy\, ds,$$

$$\psi_t(t) = \int e_t(x,t-s;y)v_0 dy\, ds + \int_0^t \int e_t(x,t-s;y)Q_x dy\, ds,$$

where $Q = \mathcal{O}(|(v, v_x, \psi_t)|^2)$ and $\tilde{G}$, e_t decay at Gaussian rate.

Model scalar problem. $v_t - v_{xx} = (v^2)_x$, decay at Gaussian rate.

(Derivative loss treated by "damping-type" nonlinear energy estimate $\partial_t \|v\|_{H^4}^2 \le -C^{-1}|v\|_{H^4}^2 + C\|v\|_{L^2}^2$, $\Rightarrow$ H^4 slaved to L^2: generalization of Kawashima estimates.)

Comments

Tehnical issue is *lack of spectral gap* of L, larger problem. Examples:. Front/pulse stability for Cahn–Hilliard, modulation of periodic wave trains for Allen–Cahn, reaction–diffusion. Corresponding issues for bifurcation, invariant manifolds.

All amenable to our techniques. Evans condition replaces, augments spectral stability conditions of standard case (gap).

IV. Verification of stability criterion.

Small amplitude Lax shocks. Energy estimates [HZ].

Intermediate amplitudes. Numerical Evans function analysis (problem independent) [Brin1998,Humpherys-Z2006].

Large amplitudes or parameters. Asymptotic ODE (same tools used for Green function bounds!) [Plaza-Z2004,HLZ2009,...].*

Rescale
$(x, t, \tau, u, a_0) \to (-\varepsilon s(x - st), \varepsilon s^2 t, \tau/\varepsilon, -u/(\varepsilon s), a_0 \varepsilon^{-\gamma-1} s^{-2})$,
with ε so that $0 < \tau_+ < \tau_- = 1$, get

$$\begin{aligned} \tau_t + v_x - u_x &= 0, \\ u_t + u_x + (a\tau^{-\gamma})_x &= \left(\frac{u_x}{\tau}\right)_x, \end{aligned} \tag{3}$$

where $a = -\frac{\tau_+ - 1}{\tau_+^{-\gamma} - 1} = \tau_+^\gamma \frac{1 - \tau_+}{1 - \tau_+^\gamma} \to 0$ as $\tau_+ \to 0$.

High-Mach $\sim$ pressureless gas limit...

Winding number computation

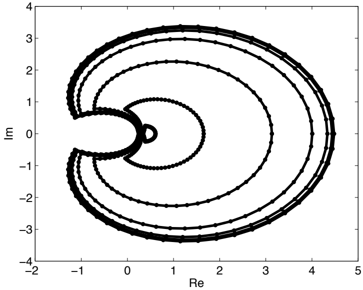


Figure: Convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas, $\gamma = 5/3$. The contours depicted, going from inner to outer, are images of the semicircle under D for $v_+ =$1e-1,1e-2,1e-3,1e-4,1e-5,1e-6. The outermost contour is the image under D^0, which is nearly indistinguishable from the image for $v_+ =$1e-6.

The large amplitude limit

Theorem (HLZ)

For λ in any compact subset of $\Re\lambda \geq 0$, $D(\lambda)$ converges uniformly to $D^0(\lambda)$ as $v_+ \to 0$.

Lemma (HLZ)

The limiting function D^0 is nonzero on $\Re\lambda \geq 0$.

Corollary

For any $\gamma \geq 1$, isentropic Navier–Stokes shocks are stable in the strong shock limit, i.e., for v_+ sufficiently small.

Case study: MHD

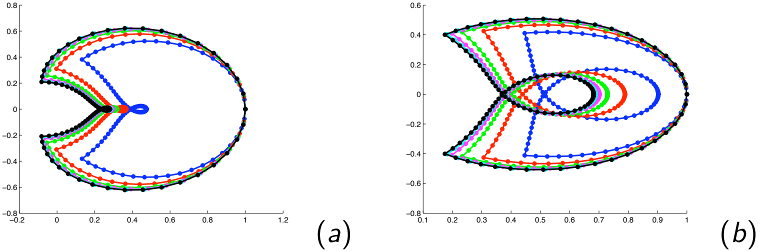


Figure: for a Lax 2-shock in two-rest point configuration in the $a \to 0$ limit, $a = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$. where $a = 10^{-8}$ corresponds to Mach number $\approx 10,954$. Convergence of contours appears to occur at $a \sim 10^{-6}$. or Mach number $\approx 1,095$. In Figure (b), for the same sequence of a-values, we display the images under the transverse Evans function, again suggestive of convergence.

V. Discussion and ongoing investigations.

EXTENSIONS: Semi-discrete and time- and space-periodic waves. Multi-dimensional planar waves. Full, nonisentropic case. Relaxation, combustion, reaction diffusion, boundary layers.

Modulation of periodic traveling wave-trains (formal "Whitham" equation predicts modulation by parameters obeying hyperbolic/hyperbolic-parabolic system).

CURRENT DIRECTIONS.

- Numerical proof.
- Viscosity, entropy, and stability– Hopf bifurcation?
- Multi-dimensional flow with general geometry.