Stability of viscous shock waves and beyond

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Lecture 1b, continuing: Classical approach

SPECTRAL STABILITY $\Rightarrow$ LINEARIZED STABILITY/ESTIMATES $\Rightarrow$ NONLINEAR STABILITY/ESTIMATES.

DETERMINATION OF SPECTRAL STABILITY then studied in its own right...

Example: finite dimensions (ODE) with spectral gap

(LYAPUNOV'S THEOREM)

$dx/dt = Ax + N(x)$, A linear, $|N(x)| \leq C|x|^2$.

Variation of constants/Duhamel principle:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}N(x(s))ds.$$

Exponential stability $\Re\sigma(A) < \eta \Rightarrow |e^{At}| \leq Ce^{-\eta t}$, $\eta > 0$, for $t > 0$

$\Rightarrow$ nonlinear exponential stability $|x(t)| \leq Ce^{-\eta t}|x_0|$, same rate.
Proof: contraction mapping in time-weighted norm

$$|f|_\eta := \sup_{[0,\infty)} |f(t)e^{\eta t}|.$$

Example: constant coefficients (perturbed heat equation)

$$u_t - Lu = (u^2)_x, \qquad L = \partial_x^2.$$

(Burgers equation, stability of $u \equiv 0$ solution.)

Fourier transform shows spectra $\sigma(L)$ now continuous, negative real axis $(-\infty, 0]$- *no spectral gap*, and no exponential decay.

Hausdorff-Young inequality gives

$$\|e^{Lt}f\|_p \le \|e^{-k^2t}\hat f\|_q \le \|e^{-k^2t}\|_q \|\hat f\|_\infty \le Ct^{-\frac{1}{2}(1-1/p)}\|f\|_{L^1},$$

$q \le 2 \le p$, $1/p + 1/q = 1$. (Direct evaluation of the Green kernel yields same bound for $1 \le p \le \infty$.)
Similarly, derivative bound $\|e^{Lt}\partial_x f\|_p \le Ct^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|f\|_{L^1}$, and same-norm bounds

$$\|e^{Lt}f\|_p \le C\|f\|_p, \qquad \|e^{Lt}\partial_x f\|_p \le Ct^{-\frac{1}{2}}\|f\|_p.$$

Time-algebraic estimate ($L^1 \cap L^\infty \to L^p$)

Variation of constants/Duhamel principle:

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}\partial_x(u(s)^2)ds.$$

Define now

$$\zeta(t) := \sup_{0\le s\le t,\, 1\le p\le\infty} |u|_{L^p}(s)(1+t)^{\frac{1}{2}(1-1/p)}. \tag{1}$$

Lemma

For all $t \ge 0$ for which $\zeta(t)$ is finite, some $C > 0$, and $E_0 := |u_0|_{L^1\cap L^\infty} \ll 1$,

$$\zeta(t) \le C(E_0 + \zeta(t)^2). \tag{2}$$

Proof

Applying our estimates, we have

$$
\begin{aligned}
|u(\cdot,t)|_{L^p(x)} \le{}& \left|e^{Lt}u_0\right|_{L^p(x)} \\
&+ \left|\int_0^{t/2} e^{L(t-s)}\partial_x(u(s))^2 ds\right|_{L^p(x)} \\
&+ \left|\int_{t/2}^{t} e^{L(t-s)}\partial_x(u(s))^2 ds\right|_{L^p(x)} \\
\le{}& C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 \\
&+ C\zeta(t)^2 \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-1/2}(1+s)^{-\frac{1}{2}} ds \\
&+ C\zeta(t)^2 \int_{t/2}^{t} (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} ds \\
\le{}& C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)}.
\end{aligned}
\tag{3}
$$

Nonlinear stability result

Using $\zeta(t) \le C(E_0 + \zeta(t)^2)$ and short-time existence/continuity of $L^1 \cap L^p$ in time, a "continuous induction" argument gives $\zeta(t) \le 2CE_0$ for all time, so long as $E_0 < \frac{1}{4C^2}$.
Boundedness of the time-algebraically weighted norm ζ gives the desired nonlinear decay estimate (Gaussian rate)

$$\|u(t)\|_p \le 2C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1 \cap L^\infty}.$$

Rmks. 1. Compare the radius needed for Lyapunov contraction argument. 2. Note that we need precisely the Gaussian decay rate to close the argument.

This is the model for all our diffusive stability arguments!

Example: Burgers equation (shock)

Finally, consider a (WLOG) standing viscous shock solution $\bar{u}(x) = -\tanh(x/2)$ of Burgers equation, $u_t + (u^2/2)_x = u_{xx}$.

Linearized equation:

$$v_t - Lv = (u^2/2)_x, \qquad Lv := (-\partial_x \bar{u} + \partial_x^2)v.$$

Explicit Green kernel for linearized Burgers equation

Linearized Hopf–Cole transformation gives

$$e^{Lt}f = \int_{-\infty}^{+\infty} G(x,t;y)f(y)dy, \tag{4}$$

where $G(x,t;y) := e^{Lt}\delta_y(x)$ is

$$\bar{u}'(x)\Big(\frac{1}{2}\Big)\Big(\mathrm{errfn}\big(\frac{x-y-t}{\sqrt{4t}}\big) - \mathrm{errfn}\big(\frac{x-y+t}{\sqrt{4t}}\big)\Big)$$
$$+\Big(\Big(\frac{e^{-\frac{x}{2}}}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\Big)\frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}} + \Big(\frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\Big)\frac{e^{-\frac{(x-y+t)^2}{4t}}}{\sqrt{4\pi t}}\Big) \tag{5}$$

and $\mathrm{errfn}(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-\xi^2}d\xi$.

Decomposition of the Green kernel

Decompose now

$$G(x,t;y) := E(x,t;y) + \tilde{G}(x,t;y), \tag{6}$$

where $E(x,t;y) := \bar{u}'(x)e(y,t)$ with

$$e(y,t) := \Big(\frac{1}{2}\Big)\Big(\mathrm{errfn}\big(\frac{-y-t}{\sqrt{4t}}\big) - \mathrm{errfn}\big(\frac{-y+t}{\sqrt{4t}}\big)\Big), \tag{7}$$

and $\tilde{G}$ is the remaining ("Good," or "Gaussian") part.

Key Obs. Solution kernels e_t and $\tilde{G}$ satisfy Gaussian decay rates!

Nonlinear iteration

Letting $\tilde{u}$ be a second solution of Burgers eqn., define the perturbation

$$u(x,t) := \tilde{u}(x + \alpha(t), t) - \bar{u}(x) \tag{8}$$

as the difference between a *translate of* $\tilde{u}$ and the background wave $\bar{u}$, where the translation $\alpha(t)$ is to be determined later. This yields after a brief computation the *perturbation equation*

$$u_t - Lu = N(u)_x + \dot{\alpha}(t)(\bar{u}_x + u_x), \tag{9}$$

where $Lu := u_{xx} - (a(x)u)_x$ is the linearization of $u_{xx} - f(u)_x$ about solution $\bar{u} = -\tanh(x/2)$, $a(x) := df(\bar{u})(x) = \bar{u}(x)$, and $N(u) := -u^2/2$ is the same quadratic order remainder as in the constant-coefficient case.

Extraction of the phase

Recalling that $\bar{u}'(x)$ is a stationary solution of the linearized equations $u_t = Lu$, so that $L\bar{u}_x = 0$, or

$$\int_{-\infty}^{\infty} G(x,t;y)\bar{u}_x(y)dy = e^{Lt}\bar{u}_x(x) = \bar{u}_x(x),$$

we have, applying Duhamel's principle to (9),

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G(x,t;y)u_0(y)\,dy \\ &- \int_0^t \int_{-\infty}^{\infty} G_y(x,t-s;y)(N(u)+\dot{\alpha}u)(y,s)\,dy\,ds \\ &+ (\alpha(t)-\alpha(0))\bar{u}'(x). \end{aligned} \tag{10}$$

Extraction of the phase

Defining α implicitly as

$$\alpha(t) = -\int_{-\infty}^{\infty} e(y,t)u_0(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(N(u)+\dot{\alpha}\,u)(y,s)dyds, \tag{11}$$

where e is defined as in (7), and substituting in (10) the decomposition $G = \bar{u}'(x)e + \tilde{G}$ above, we obtain the *integral representation*

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{G}(x,t;y)u_0(y)\,dy - \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x,t-s;y)(N(u)+\dot{\alpha}u)(y,s)dy\,ds, \tag{12}$$

and, differentiating (11) with respect to t, and observing that $e_y(y,s) \to 0$ as $s \to 0$,

Extraction of the phase

$$\dot\alpha(t) = -\int_{-\infty}^{\infty} e_t(y,t)u_0(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(u)+\dot\alpha u)(y,s)\,dy\,ds. \tag{13}$$

(Note: in obtaining (12), we have used the fact that $e(y,0)\equiv 0$ to conclude that $\alpha(0)=0$.)

Closed system in v, $\dot\alpha$!

Nonlinear iteration

Define

$$\zeta(t) := \sup_{0\le s\le t,\, 1\le p\le\infty} \Big(|u|_{L^p}(s)(1+t)^{\frac{1}{2}(1-1/p)} + |\dot\alpha(s)|(1+s)^{1/2} \Big). \qquad (14)$$

Lemma

For all $t \ge 0$ for which $\zeta(t)$ is finite, some $C > 0$, and $E_0 := |u_0|_{L^1\cap L^\infty}$,

$$\zeta(t) \le C(E_0 + \zeta(t)^2). \qquad (15)$$

With the established bounds on $\tilde G$ and e, the proof of (15) is almost identical to that of (2) in the constant-coefficient case.

Conclusion

Theorem

Viscous shock solutions $\bar{u}(x)$ of Burgers eqn. are nonlinearly stable in $L^1 \cap L^\infty$ and nonlinearly orbitally asymptotically stable in L^p, $p > 1$, with respect to initial perturbations u_0 that are sufficiently small in $L^1 \cap L^\infty$. More precisely, for some $C > 0$ and $\alpha \in W^{1,\infty}(t)$,

$$\begin{aligned} |\tilde{u} - \bar{u}(\cdot - \alpha)|_{L^p}(t) &\leq C(1+t)^{-\frac{1}{2}(1-1/p)} |\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}|_{t=0}, \\ |\dot{\alpha}(t)| &\leq C(1+t)^{-\frac{1}{2}} |\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}|_{t=0}, \\ |\alpha(t)| &\leq C |\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}|_{t=0}, \\ |\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}(t) &\leq C |\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}|_{t=0}, \end{aligned} \tag{16}$$

for all $t \geq 0$, $1 \leq p \leq \infty$, for solutions $\tilde{u}$ with $|\tilde{u} - \bar{u}|_{L^1 \cap L^\infty}|_{t=0}$ sufficiently small.

To follow

NEXT: Deriving corresponding linearized (Green function) bounds for *general* viscous shock waves based on spectra of the linearized operator about the wave...