Stability of viscous shock waves and beyond

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Sponsored by NSF Grants no. DMS-0300487 and DMS-0801745

Short course, IHP: Lecture 2a

Spectra of differential operators and Inverse Laplace transform formulae

TARGET: Linearized operator

$$Lu = -(Au_x) + (Bu_x)_x$$

about a viscous shock wave, asymptotically constant-coefficient with exponential rate as $x \to \pm\infty$.

GOAL: Relate spectra to estimates for linearized evolution equation

$$u_t = LV.$$

General method: Inverse Laplace tranform representation and analytic function theory.

(Laplace transform: $\hat{f}(\lambda) := \int_0 e^{-\lambda t} f(t) dt$.

Setting $\lambda = ik + \gamma$, can consider as Fourier transform $\int_{-\infty}^{+\infty} e^{ikt}\big(e^{-\gamma t} f(t) H(t)\big) dt$ of $f(t)e^{-\gamma t}$ times Heaviside fn. H.

Implications: (i) convergent when $|f(t)| \leq Ce^{\gamma_0 t}|$ and $\gamma > \gamma_0$.
(ii) (from Fourier version, P.V. to accomodate Heaviside)

INVERSE LAPLACE TRANSFORM FORMULA:

$$f(t) = P.V. \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \hat{f}(\lambda) d\lambda.$$

Some (very) basic properties

Evaluation by Residue theory, example

$$P.V.\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}/\lambda d\lambda = H(t),$$

Heaviside function, for $\gamma > 0$. (Move contour, decreasing γ to $-\infty$, residue at $\lambda = 0$.)

We'll need this (only!), plus (later) fact that Fourier inverse of $e^{-k^2\alpha}$ is $e^{-x^2/4b}/\sqrt{4\pi\alpha}$, formula for heat kernel.

Also, differentiation formula (for f differentiable at 0^+):

$$\widehat{\partial_t f} = \lambda\hat{f} - f(0).$$

(From definition, integrating by parts.)

ODE: Solution formula via ILT

For ODE $dx/dt = Ax$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, constant, seek matrix-valued solution operator $X(0) = \mathrm{Id}$, $dX/dt = AX$. As $|X(t)| \le e^{|A|t}$, Laplace tranform is defined for $\gamma > |A|$.

By derivative formula $(\lambda - A)\hat{X} = \mathrm{Id}$, hence

$$\hat{X}(\lambda) = \mathrm{Resolvent(A)} := (\lambda - A)^{-1}.$$

Consequence is **Inverse Laplace transform representation**:

$$X(t) = P.V. \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - A)^{-1} d\lambda. \qquad \text{(ILT)}$$

Extracting bounds from spectrum

The resolvent $(\lambda - A)^{-1}$ is holomorphic, with poles at eigenvalues of A.

Move contour to $\gamma >$ max. $\Re\sigma(A)$ in ILT formula

$$X(t) = P.V. \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda - A)^{-1} d\lambda.$$

Next, use **RESOLVENT IDENTITY** (from definition):

$$(\lambda - A)^{-1} = (A/\lambda)(\lambda - A)^{-1} + \mathrm{Id}/\lambda$$

to split as abs. conv. $|\int e^{\lambda t}(A/\lambda)(\lambda - A)^{-1}| \sim e^{\gamma t} \int |\lambda|^{-2}$ plus explicit $P.V. \int e^{\lambda t}/\lambda = H(\gamma)$ for $t > 0$.

In either case, **EXPONENTIAL BOUND**: $|X(t)| \leq Ce^{\gamma t}$.

Remark. Moving contour to real part $-\infty$ leaves summation over eigenvalues, *spectral decomposition formula* (included in ILT).

II. Unbounded operators

Let L be a (possibly unbounded) operator from domain $D(L) \subset X \to$ Banach space X that is *densely defined* ($D(L)$ dense in X) and *closed* ($x_n \to x$ and $Lx_n \to y \Rightarrow x \in D(L)$ and $Lx = y$).

Ex. Show that $D(L)$ is a Banach space under the canonical norm $|x|_D := |x| + |Lx|$ iff L is closed.

The spectrum $\sigma(L)$ is defined as λ such that there does not exist a bounded resolvent $R(L,\lambda) = (\lambda - L)^{-1} : X \to D(L)$. (Bounded in X-norm $|\cdot|$, or, by bounded inverse thm., from X- to D-norm.) Complement is *resolvent set* $\rho(L)$. Eigenvalues are *point spectrum*, rest = *essential spectrum*.

Example: $L = \partial_x^2 + A(x)\partial_x$, $X = L^2$, $D(L) = H^2$. Closed because $|x| + |Lx| \sim |x|_{H^2}$.

Properties

- $R(L,\lambda)$ is analytic on $\rho(L)$ (in particular, $\rho(L)$ an open set). *pf.* $(\lambda - L)^{-1} = (\lambda_0 - L)^{-1}(\mathrm{Id} - (\lambda - \lambda_0)(\lambda_0 - L)^{-1})^{-1}$ plus convergence of Neumann expansion.

- If $\rho(L) \neq \emptyset$, then $D(L^n)$ (defined as x s.t. $Lx \in D(L^{n-1})$) is dense in X for all n.

NOW, suppose that $|R(L,\lambda)| \leq C$ for all $\Re\lambda \geq \gamma$ (in particular, all spectra to the left of $\Re\lambda = \gamma$)– typical justification = energy estimates, or analyticity plus large $|\lambda|$/semiclassical limit analysis (WKB)- and consider ILT for $x \in D(L^2)$:

$$S(t) = P.V.\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda - L)^{-1} x d\lambda.$$

Validation

1. **Convergence:** Using resolvent identity $(\lambda - L)^{-1} = (L/\lambda)(\lambda - L)^{-1} + \mathrm{Id}/\lambda$, split into explicitly convergent $\int e^{\lambda t}/\lambda = H(\gamma)$ part and $\int e^{\lambda t}(L/\lambda)(\lambda - L)^{-1} d\lambda$. Applying resolvent identity to this piece, get abs. convergent

$$\int e^{\lambda t}(L/\lambda)(L/\lambda(\lambda - L)^{-1} + 1/\lambda)x d\lambda \sim e^{\gamma t} \int 1/\lambda^2 d\lambda.$$

Moreover, get exponential bound with loss of regularity:

$$|S(t)x| \leq Ce^{\gamma t}(|L^2 x| + |Lx| + |x|).$$

Validation

2. **Equation:** For $x \in D(L^3)$, we have convergence of $L(ILT)$ and also $(d/dt)ILT$ (corresponding to additional λ factor in integrand). But also $(d/dt - L)$ annihilates the integrand. By closure, $L(\text{limit})=$ limit of L applied to finite sums, so $(d/dt - L)S(t)x = 0$.
2. **Data:** By the resolvent identity expansion used for convergence,

$$\begin{aligned}(S(t) - \mathrm{Id})x &= P.V.\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(L/\lambda)(\lambda - L)^{-1}x d\lambda.\\ &= P.V.\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(L/\lambda)\big((L/\lambda)(\lambda - L)^{-1} + 1/\lambda\big)x d\lambda,\end{aligned} \tag{1}$$

which is bounded by $Ce^{\gamma t}\int 1/(\gamma^2 + k^2)dk \to 0$ as $t \to 0^+$ moving $\gamma \sim 1/t$. (Here, used no spectra to right of γ...)

So, data taken on, $\lim_{t\to 0^+} S(t)x = x$, for $x \in D(L^2)$.

Validation: final step

We have justified the ILT as solution formula for strong (classical) solutions in high regularity class. If, then, we can obtain uniform bounds from $|x| \to |S(t)x|$ on these strong solutions, *no matter how poor*, then we can extend by continuity (using again closure of L) to a solution operator on all of X, corresponding to a mild, or weak solution (slightly different, both preserved under limits).

Closing the loop: Gronwall and damping bounds

The above $\sim$ C_0-semigroup arguments [Pazy], but lacking $|x| \to |S(t)x|$ bound of semigroup. In practice, semigroup property typically verified by *Lumer-Phillips Theorem*, energy estimate on resolvent. In Hilbert space:

$$\langle x, Lx\rangle \le C|x|^2$$

(In fact, $C = 1$, or else further estimates on powers of $(L - \gamma_0)$).

But, this gives desired uniform $|x(t)|$ bound on classical solutions $x(0) \in D(L^3)$ via

$$(d/dt)\frac{1}{2}\langle x(t), x(t)\rangle = \langle x(t), Lx(t)\rangle \le C\langle x, x\rangle,$$

verifying ILT (and C_0 semigroup property for its closure).

Damping estimate and exponential decay

For the equations we consider in these lectures, a stronger, *damping estimate* holds:

$$\sum_{j=0}^{s} \langle L^j x, L L^j x\rangle \le -2\eta |x|^2_{D(L^s)} + C|x|^2, \qquad \eta > 0,$$

or

$$(d/dt)\frac{1}{2}|x(t)|^2_{D(L^s)} \le -2\eta |x(t)|^2_{D(L^s)} + C|x|^2,$$

giving (slaved) Gronwall estimate:

$$|x(t)|_{D(L^s)} \le e^{-\eta t}|x(0)|_{D(L^s)} + C\int_0^t e^{-\eta(t-s)}|x(s)|ds. \qquad (2)$$

Together with $|x(t)| \le Ce^{-\theta t}|x(0)|_{D(L^s)}$ **bound coming from ILT, recovers exponential decay.** ($\sim$ Prüss Theorem):

$$|x(t)|_{D(L^s)} \le Ce^{-\min\{\eta,\theta\}t}|x(0)|_{D(L^s)}.$$

Conclusions

We have for the class of equations considered obtained the ILT solution formula and C_0-semigroup properties directly, including a Prüss-type theorem giving exponential decay. We'll see damping estimates again at nonlinear level, used to avoid delicate maximal regularity issues for quasilinear hyperbolic-parabolic case.

EXAMPLE, damping: For $L = \partial_x^2 + A(x)\partial_x + B(x)$, differentiate $u_t = Lu$ s times and take energy estimate, to obtain $|u(t)|_{H^s}^2 \leq -\theta|\partial_x^{s+1}u|^2 - C|u|_{H^s}$. Using Sobolev interpolation,

$$|f|_{H^s} \leq (1/C)|f|_{H^{s+1}} + C|f|_{H^{s-1}}$$

(Young's inequality in Fourier space), get damping with rate $\eta > 0$ arbitrarily large (so full "Prüss" result).

Examples/comments

Constant coefficients: $L = \partial_x^2 - A\partial_x$, $X = L^2$. By Fourier transform, spectra consists of *dispersion curves*

$$\{\lambda :; \lambda = \lambda_j(k) \in \sigma(-k^2 - iAk),\ k \in \mathbb{R}\}.$$

(where $(\lambda - L) \sim (\lambda - (-k^2 - iAk))$ not invertible on L^2.) Moreover, entirely *essential spectrum* (generalized eigenfunctions).

Moving $\gamma \to -\infty$ in ILT gives line integrals of $e^{\lambda t}$ times jump in $R(L, \lambda)$ across dispersion curves = spectral projection.

RECOVERS FOURIER INVERSE SOLUTION FORMULA $S(t)(f) = \mathcal{F}^{-1} \int \sum_j e^{\lambda_j(k)t} P_j(k) \hat{f}\, dk$, $P_j(k)$ the eigenprojection associated with $\lambda_j(k)$.

Examples/comments

Self-adjoint operators: Spectra confined to real axis, in general combination of point and essential spectrum. "Wrapping" contour around negative real axis gives combination of residues (point support, eigenprojections) and jumps (line measures, generalized eigenprojection),

RECOVERS GENERALIZED SPECTRAL DECOMPOSITION.

Remark. We shall see, for nonself-adjoint operators, things can be more complicated (two-dimensional measure, effectively).

Can think of ILT as generalization/well-conditioned version of spectral decomposition. A primary tool in several disciplines...