Stability of viscous shock waves and beyond

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Sponsored by NSF Grants no. DMS-0300487 and DMS-0801745

Short course, IHP: Lecture 3a

Linearized bounds I: Energy estimates and high-frequency solution bounds

GOAL: By a circle of ideas centered around "Kawashima-type" energy estimates, for **general** hyperbolic-parabolic system and linearized operator $Lu = -(Au_x) + (Bu_x)_x$ about a viscous shock:

- Estimate $\sigma_{ess}(L)$ via dispersion curves of limiting operators

$$L_\pm = (-A\partial_x + B\partial_x^2)_\pm.$$

- Estimate $|(\lambda - L)^{-1}|_{H^s} \leq C$ for $|\lambda| \geq R >> 1$, $\Re\lambda \geq -\eta$, $\eta > 0$.
- Obtain nonlinear damping estimate $\approx$

$$(d/dt)\frac{1}{2}|u(t)|^2_{H^s} \leq -\eta|u(t)|^2_{H^s} + C|u|^2_{L^2}, \qquad \eta > 0.$$

I. Equations and assumptions

Consider a viscous shock solution:

$$U(x,t) = \bar{U}(x - st), \qquad \lim_{z\to\pm\infty} \bar{U}(z) = U_\pm, \tag{1}$$

of a hyperbolic–parabolic system of conservation laws:

$$U_t + F(U)_x = (B(U)U_x)_x, \qquad x \in \mathbb{R},\ U,\ F \in \mathbb{R}^n,\ B \in \mathbb{R}^{n\times n}. \tag{2}$$

Profile $\bar{U}$ satisfies the traveling-wave ODE:

$$B(U)U' = F(U) - F(U_-) - s(U - U_-). \tag{3}$$

In particular, the condition that $U_\pm$ be rest points implies the *Rankine–Hugoniot condtions*:

$$F(U_+) - F(U_-) = s(U_+ - U_-) = 0 \tag{4}$$

of inviscid shock theory.

Structural assumptions

(A1) $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad b$ nonsingular, where $U \in \mathbb{R}^n$, $U_1 \in \mathbb{R}^{n-r}$, $U_2 \in \mathbb{R}^r$, and $b \in \mathbb{R}^{r\times r}$. (In the examples, $F_1(U) = A_{11}U_1 + A_{12}(U_2)$, *linear*; important for Hopf bifurcation, conditional stability analyses.)

(A2) There exists a smooth, positive definite matrix field $A^0(U)$, without loss of generality block-diagonal, such that $A^0_{11}A_{11}$ is symmetric definite (+ or -), $A^0_{22}b$ is positive definite (not necessarily symmetric), and $(A^0A)_\pm$ is symmetric. (In the examples, $A^0_{11} = \mathrm{Id}$.)

(A3) No eigenvector of $A_\pm$ lies in $\mathrm{Ker}B_\pm$ *(Kawashima's genuine coupling condition (GC)).* (In the examples, $A_{11} = \alpha\mathrm{Id}$, corresponding to simple transport along fluid particle paths, with genuine coupling equivalent to A_{12} full rank.)

Recall, example: 2D isentropic MHD

$$
\begin{aligned}
\tau_t - s\tau_x - u_{1,x} &= 0,\\
(\tau B_2)_t - s(\tau B_2)_x - (B_1^* u_2)_x &= 0,\\
u_{1t} - su_{1x} + (p(\tau) + \left(\frac{B_2^2}{2\mu_0}\right)_x &= \left(\frac{(2\mu+\eta)u_{1x}}{\tau}\right)_x,\\
u_{2t} - su_{2x} - \left(\frac{B_1^* B_2}{\mu_0}\right)_x &= \left(\frac{\mu u_{2x}}{\tau}\right)_x,
\end{aligned}
$$

$\tau \in R$, $u, B \in R^2$, $B_1^* \equiv \text{constant}$ ($\Leftrightarrow \nabla \cdot B = 0$), $\mu, \eta > 0$, $p(\tau) = c\tau^{-\gamma}$, $\gamma > 1$.

Entropy $\eta = \int_\tau^{+\infty} p(z)dz + \frac{|u|^2}{2} + \frac{\tau |B|^2}{2\mu_0}$; hyperbolic wherever p monotone $\Leftrightarrow$ η convex ($\Rightarrow$ symmetrizablility). Shock speed $s \neq 0$ ($\Rightarrow$ genuine coupling (GC)). $A^0 = \mathrm{Id}$, $A_{11} = -s\mathrm{Id} < 0$.

Technical hypotheses

(H1) $F, B \in C^k$, $k \geq 5$.

(H2) $\sigma(A_\pm)$ (necessarily real) is simple, $\neq s$. (Generically holds.)

(H3) Considered as connecting orbits of (3), $\bar{U}$ is a *transversal connection*, $\Rightarrow$ "Lax-type" shock wave, local uniqueness up to translation. ("Standard" type shock familiar from gas dynamics.)

Remark. (H2) with (A1)–(A3) implies exponential decay to endstates $U_\pm$ of $\bar{U}(x)$ as $x \to \pm\infty$ [Handbook, on reserve].

Functional-analytic setting

Linearized operator $LU = -(AU)_x + (\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} U_x)_x$ is closed, densely defined on domain $D(L) = \begin{pmatrix} H^{s+1} \\ H^{s+2} \end{pmatrix} \subset H^s$, any s for which A, B sufficiently smooth. *(Here, use (A2),* $\det A_{11} \neq 0$*...)*

By exponential decay, $\sigma(L)$ has only eigenvalues to the right of $\sigma(L_\pm)$, hence **bound** $\Re\sigma(L_\pm) < \gamma$ plus **a priori bound**

$$|(\lambda - L)u|_{H^s} \geq (1/C)|u|_{H^s}, \qquad \Re\lambda \geq \gamma,$$

is sufficient to justify ILT solution formula for $U_t = LU$:

$$S(t)U = P.V.\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda - L)^{-1} U d\lambda, \qquad U \in D(L^2).$$

II. Essential spectrum boundaries: Dispersion curves of $(-iAk - k^2 B)_\pm$ (equal to $\sigma(L_\pm)$)

Hyperbolic-parabolic smoothing:

Lemma (Kawashima-Shizuta)

Assuming A^0, A, B symmetric, $A^0 > 0$, and $B \geq 0$, the genuine coupling condition (GC) is equivalent to either of:

(K1) There exists a smooth skew-symmetric matrix function $K(A^0, A, B)$ such that

$$\mathrm{Re}\ \left(K(A^0)^{-1}A + B\right)(U) > 0. \tag{5}$$

(K2) For some $\theta > 0$, all $k \in \mathbb{R}$,

$$\mathrm{Re}\ \sigma(-ik(A^0)^{-1}A - k^2(A^0)^{-1}B) \leq -\theta \frac{k^2}{(1+k^2}. \tag{6}$$

Proof

Without loss of generality, $A^0 = \mathrm{Id}$, A=diagonal.

(GC)⇒(K1). $\Re(KA) := \frac{1}{2}(KA + AK^T)$ vanishes on diagonal, j-k entry $(a_j - a_k)K_{jk}$ can be chosen to cancel off-diagonal terms of (symmetric) matrix B. Hence, $\Re(KA + B) = \mathrm{diag}\, B$, nonvanishing by (GC), and nonnegative by semidefiniteness of B.

(K1)⇒(K2). Writing $\lambda U + AikU = k^2 BU$, get

$$\begin{aligned}(\Re\lambda)\langle (Ck^2 + ikK + C)U, U\rangle &= \Re\langle (CA^0k^2 + ikK + CA^0)U, \lambda U\rangle \\ &= \Re\langle (CA^0k^2 + ikK + CA^0)U, (Aik - k^2B)U \\ &\leq -\langle ikU, \Re(A^0B + KA)ikU\rangle \\ &\leq -\theta k^2 |U|^2. \end{aligned} \tag{7}$$

(K2)⇒(GC) Spectral perturbation expansion about $k = 0$ gives $\lambda_j(k) = ika_j - B_{jj}k^2$, yielding $B_{jj} > 0$ and (GC).

Bounds on essential spectra

Corollary

The essential spectra $\sigma_{ess}(L_\pm)$ and (therefore) $\sigma_{ess}(L)$ lie within

$$\Re\lambda \le -\theta \frac{|\Im\lambda|^2}{1+|\Im\lambda|^2}, \qquad \theta > 0.$$

Remark. In particular, for $|\lambda| >$ any $c > 0$, $\Re\lambda \ge -\theta(c)$ is contained in the resolvent set $\rho(L)$, $\theta(c) > 0$.

III. A Priori resolvent estimate for $|\lambda|$ large

Let us first replace the Fourier computation of (7) with its equivalent spatial-domain version, replacing *ik* with ∂_x.

$$\begin{aligned}(\Re\lambda)\langle(-C\partial_x^2 + K\partial_x + C)U, U\rangle &= \Re\langle(-CA^0\partial_x^2 + K\partial_x + CA^0)U, \lambda U\rangle \\ &= \Re\langle(-CA^0\partial_x^2 + K\partial_x + CA^0)U, (-\partial_x A - \\ &\leq -\langle\partial_x U, \Re(A^0B + KA)\partial_x U\rangle \\ &\leq -\theta|\partial_x U|^2.\end{aligned} \tag{8}$$

Weak shocks (slowly varying coeffs.): Setting $LU = f$, and repeating the above, absorbing commutators and adding $\langle U, f\rangle$ terms, get:

$$\Re\lambda|U|_{H^1}|^2 \leq -\theta|U_x|_{L^2}^2 + C|U|_{L^2}^2 + |f|_{H^1}^2. \tag{9}$$

Large λ estimate

Combining $\Re\lambda$ estimate

$$\Re\lambda|U|_{H^1}|^2 \leq -\theta(|U_{2,xx}|_{L^2}^2 + |U_x|_{L^2}^2) + C|U|_{L^2}^2 + |f|_{H^1}^2$$

with easy $|\lambda|$ estimate:

$$|\lambda||U|_{L^2}^2 \leq |\langle U, (-\partial_x A + \partial_x B \partial_x)U + f\rangle| \leq C|U|_{H^1}^2,$$

or $|U|_{L^2}^2 \leq C|\lambda|^{-1}(|U|_{H^1}^2 + |f|_{L^2}^2)$, get finally

$$(\Re\lambda + \theta_1)|U_{H^1}^2 \leq |f|_{H^1}^2,$$

verifying for $|\lambda| \gg 1$ the desired a priori resolvent bound for $\Re\lambda \geq -\theta_1/2$.

Strong (large amplitude) shocks

Now $\Re(A^0 B + KA)$ is not definite (neither symmetrizability nor (GC) assumed away from endstates...). To deal with shock layer $x \in [-R, R]$, use "Goodman-style" estimate, exponential weight $\sim |\partial_x A_{11}||A_{11}^{-1}|$ to gain "good term" $\sim -|U_x||U_1|^2_{H^1}$ on RHS, good *away from* endstates $x \to \pm\infty$, **recover same estimate** (9).

Details omitted (see [Handbook], on reserve. Heuristic principle is that $\det A_{11} \neq 0$ means hyperbolic signals have nonzero speed relative to shock, pass quickly out of $[-R, R]$, as recorded by decay with respect to exponential weight- goes back to Goodman's handling of transverse modes in proof of zero-mass stability for weak shocks (roughly equivalent to spectral stability) of strictly parabolic systems.

III. High-frequency linearized bounds

Assuming that L has no eigenvalues on $\{\Re\lambda \geq 0\} \setminus \{0\}$, as we shall assume for our stability theorem, then we can move contours to decompose

$$\begin{aligned} S(t)U &= P.V.\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - L)^{-1}U d\lambda \\ &= P.V.\frac{1}{2\pi i}\Big(\int_{-\theta-i\infty}^{-\theta-i\epsilon} + \int_{-\theta+i\epsilon}^{-\theta+i\infty}\Big) e^{\lambda t}(\lambda - L)^{-1}U d\lambda \\ &\quad + \frac{1}{2\pi i}\oint_{\Gamma} e^{\lambda t}(\lambda - L)^{-1}U d\lambda \\ &=: S_{HF}(t) + S_{LF}(t), \end{aligned} \tag{10}$$

where Γ is an appropriate (finite) contour.

Result (store for later)

Proposition

Under (A1)–(A3), (H1)–(H3), and absence of nonnegative eigenvalues $\lambda \neq 0$ *of L:*

$$|S(t)f|_{H^1} \leq e^{-\theta t}|f|_{H^5} \tag{11}$$

IV. Nonlinear damping estimate

Finally, replace the Fourier computation of (7) with its full spatio-temporal equivalent

$$\begin{aligned}
\partial_t \frac{1}{2}\langle(-C\partial_x^2 + K\partial_x + C)U, U\rangle &= \langle(-CA^0\partial_x^2 + K\partial_x + CA^0)U, \partial_t U\rangle \\
&= \Re\langle(-CA^0\partial_x^2 + K\partial_x + CA^0)U, (-\partial_x A + \\
&\leq -\langle\partial_x U, \Re(A^0 B + KA)\partial_x U\rangle \\
&\leq -\theta|\partial_x U|^2.
\end{aligned} \tag{12}$$

get $\partial_t \mathcal{E}(U(t)) \leq -\theta\mathcal{E}(U(t)) + C|U(t)|_{L^2}^2$, where
$\mathcal{E}(U) := \sum_{j=0}^4 \langle(-CA^0\partial_x^2 + K\partial_x + CA^0)\partial_x^j U, \partial_t\partial_x^j U\rangle$ is $\sim |U|_{H^5}$

Conclusion (store for later)

Set

$$U(x,t) := \tilde{U}(x+\alpha(t),t) - \bar{U}(x), \tag{13}$$

here (as in sample Burgers argument) "shock location" α is to be determined later. Then (see [Handbook], or [Mascia-Z]* for details), absorbing commutator and quasilinear terms, and accomodating the effects of new $\dot{\alpha}(\bar{U}_x + U_x)$ terms in the equation, we get:

Proposition

Under (A1)–(A3), (H1)–(H3),

$$|U(t)|_{H^5}^2 \le C\Big(e^{-\theta t}|U(0)|_{H^5}^2 + \int_0^t e^{-\theta(t-s)}(|U|_{L^2}^2 + |\dot{\alpha}|^2)(s)\,ds\Big), \quad \theta > 0, \tag{14}$$

so long as $|U|_{H^5}$ remains sufficiently small.

That is:

AS IN THE LINEAR EVOLUTION SETTING, HIGHER-DERIVATIVE NORMS ARE SLAVED TO L^2.

(Thus, we will be able, later, to absorb loss of derivatives in our linear estimates/nonlinear iteration...)