Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 3b

Low-frequency bounds and basic nonlinear stability argument

At this point, we can split the ILT solution formula for $U_t = LU$:

$$S(t)U = P.V. \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda - L)^{-1} U d\lambda, \qquad U \in D(L^2),$$

into two parts: S_{HF}, the estimated time-exponentially decaying part above, and

$$\begin{aligned} S_{LF}(t)U &= \frac{1}{2\pi i} \oint_{\Gamma} e^{\lambda t} (\lambda - L)^{-1} U d\lambda \\ &= \frac{1}{2\pi i} \oint_{\Gamma} \int_{-\infty}^{\infty} G_{\lambda}(x, y) U(y) dy d\lambda, \end{aligned} \tag{1}$$

or, exchanging order of integrals by Fubini,

$$\int_{-\infty}^{\infty} G_{LF}(x, t; y) U(y) dy,$$

where $G(x, t; y)$ is a nice L^p function.

Recall HF bound...

We proved in the first lecture:

Proposition

Under (A1)–(A3), (H1)–(H3), and absence of nonnegative eigenvalues $\lambda \neq 0$ of L:

$$|S_{HF}(t)f|_{H^1} \le e^{-\theta t}|f|_{H^5} \qquad (2)$$

Also, nonlinear damping estimate

Further, setting

$$U(x,t) := \tilde{U}(x+\alpha(t),t) - \bar{U}(x), \tag{3}$$

where "shock location" α is to be determined later, we showed:

Proposition

Under (A1)–(A3), (H1)–(H3),

$$|U(t)|_{H^5}^2 \le C\Big(e^{-\theta t}|U(0)|_{H^5}^2 + \int_0^t e^{-\theta(t-s)}(|U|_{L^2}^2 + |\dot{\alpha}|^2)(s)\,ds\Big), \tag{4}$$

$\theta > 0$, *so long as* $|U|_{H^5}$ *remains sufficiently small.*

To be shown later: Low-frequency evolution bounds

Lemma

$G_{LF}(x,t;y) = \bar{U}'(x)e(y,t) + \tilde{G}(x,t;y)$, *with, for* $y<0$

$$e(y,t) := \frac{1}{2}\sum_{a_j^- > 0}\Big(errfn(\frac{-y - a_j^- t}{\sqrt{4t}}) - errfn(\frac{-y + a_j^- t}{\sqrt{4t}})\Big), \quad (5)$$

and, for $1 \le q \le 2 \le p \le \infty$, *(Gaussian rates):*

$$\Big\|\int_{-\infty}^{+\infty} \tilde{G}(x,t;y)f(y)dy\Big\|_{L^p(x)} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)}\|f\|_{L^q\cap L^p},$$

$$\Big\|\int_{-\infty}^{+\infty} \partial_y\tilde{G}(x,t;y)f(y)dy\Big\|_{L^p(x)} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{1}{2}}\|f\|_{L^q\cap L^p},$$

$$\Big\|\int_{-\infty}^{+\infty} \partial_{x,y,t}e(x,t;y)f(y)dy\Big\|_{L^p} \le (1+t)^{-\frac{1}{2}(1/q-1/p)-\frac{(j+k)}{2}}\|f\|_{L^q},$$

COMMENTS

• Includes (and by analytic interpolation equivalent to) the bounds established for Burgers equation, corresponding to extreme points $q = 1, p$.

• Will be established by showing that $\tilde{G}$, up to negligible error, is given by a summation of Gaussians moving with characteristic speeds $\sigma(A_\pm)$ associated with first-order hyperbolic systems at $x = \pm\infty$. Exactly analogous to Burgers case (where characteristics at $x = \pm\infty$ were ∓ 1).

This will be done in the following lecture, involving a different circle of ideas. For now, temporarily **assuming these bounds**, we show how to complete the stability analysis.

Nonlinear stability theorem

Theorem

Under (A1)–(A3), (H1)–(H3), and (D), viscous shock waves $\bar U(x)$ *are nonlinearly stable in* $L^2 \cap H^5$ *and nonlinearly orbitally asymptotically stable in* $L^2 \cap H^5$*, with respect to initial perturbations* U_0 *that are sufficiently small in* $L^1 \cap H^5$*. More precisely, for some* $C > 0$ *and* $\alpha \in W^{5,\infty}(t)$*,*

$$\begin{aligned} |\tilde U - \bar U(\cdot - \alpha)|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)}|\tilde U - \bar U|_{L^1\cap L^\infty}|_{t=0}, \\ |\dot\alpha(t)| &\le C(1+t)^{-\frac{1}{2}}|\tilde U - \bar U|_{L^1\cap L^\infty}|_{t=0}, \\ |\alpha(t)| &\le C|\tilde U - \bar U|_{L^1\cap L^\infty}|_{t=0}, \\ |\tilde U - \bar U|_{L^1\cap L^\infty}(t) &\le CE_0, \\ |\tilde U - \bar U|_{L^1\cap L^\infty}|_{t=0}, \end{aligned} \tag{7}$$

for all $t \ge 0$, $2 \le p \le \infty$*, for solutions* $\tilde U$ *with* $|\tilde U - \bar U|_{L^1\cap H^5}|_{t=0}$

Proof (similar as in Burgers case)

Letting $\tilde{U}$ be a second solution, define perturbation

$$U(x,t) := \tilde{U}(x+\alpha(t),t) - \bar{U}(x) \tag{8}$$

as the difference between a translate of $\tilde{u}$ and the background wave $\bar{u}$, $\alpha(t)$ to be determined, yielding *perturbation equations*:

$$U_t - LU = N(U)_x + \dot{\alpha}(t)(\bar{U}_x + U_x), \tag{9}$$

where L is the linearized operator and $N(U) = O(U^2 + |U||U_{2,x}|)$.

Extraction of the phase

Recalling that $\bar{u}'(x)$ is a stationary solution of the linearized equations $u_t = Lu$, so that $L\bar{u}_x = 0$, or

$$S(t)\bar{U}_x = \bar{U}_x,$$

we have, applying Duhamel's principle to (9),

$$u(x,t) = S(t)U_0 + \int_0^t \int_{-\infty}^{\infty} S(t-s)\partial_x(N(U) + \dot{\alpha}U)(s)\,ds + (\alpha(t) - \alpha(0))\bar{U}'(x). \tag{10}$$

Extraction of the phase

Defining α implicitly as

$$\alpha(t) = -\int_{-\infty}^{\infty} e(y,t)U_0(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(N(U)+\dot{\alpha}\,U)(y,s)dyds \tag{11}$$

where e is defined as in (5), and substituting in (10) the decomposition $G_{LF} = \bar{u}'(x)e + \tilde{G}$ above, we obtain the *integral representation*

$$U(x,t) = \int_{-\infty}^{\infty} \tilde{G}(x,t;y)U_0(y)\,dy + S_{HF}(t)U_0 - \int_0^t \int_{-\infty}^{\infty} \tilde{G}_y(x,t-s;y)(N(U)+\dot{\alpha}U)(y,s)dy\,ds + \int_0^t S_{HF}(t-s)\partial_x(N(U)+\dot{\alpha}\,U)(y,s)ds, \tag{12}$$

Extraction of the phase

and, differentiating (11) with respect to t, and observing that $e, e_y(y,s) \to 0$ as $s \to 0$:

$$\dot\alpha(t) = -\int_{-\infty}^{\infty} e_t(y,t)U_0(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N(U)+\dot\alpha U)(y,s)\,dy\,ds. \tag{13}$$

Closed system in U, $\dot\alpha$.

Nonlinear iteration

Adapting slightly the weighting of the Burgers case, define

$$\zeta(t) := \sup_{0\le s\le t,\, 2\le p\le\infty} \Big(|U|_{L^p}(s)(1+t)^{\frac{1}{2}(1-1/p)} + |U_{H^5}(1+s)^{1/4} + |\dot\alpha(s)|(1+s)^{1/2} \Big). \tag{14}$$

Lemma

For all $t\ge 0$ for which $\zeta(t)$ is finite, some $C>0$, and $E_0 := |U_0|_{L^1\cap H^5}$,

$$\zeta(t) \le C(E_0 + \zeta(t)^2). \tag{15}$$

Nonlinear iteration: proof of the Lemma

With our bounds on $\tilde{G}$, e, and $S_H(t)$, the estimates for $\dot{\alpha}$ and $|U|_{L^p}$ of (15) go almost exactly as in the Burgers case. (Note: losses in regularity due to S_H are made up for by H^5 bound.) This controls $L^2 \cap L^\infty$ by H^5

FINAL (ADDITIONAL) STEP: to obtain the needed H^5 bound, use now the nonlinear damping estimate, which controls H^5 by L^2.

(END PROOF OF LEMMA)

Nonlinear iteration: proof of the Main Theorem

To complete the proof of the main theorem, we now appeal to short-time existence/continuous dependence results of Kawashima in H^5. These give continuity of the H^5 norm in time, hence, by Sobolev embedding, of $L^2 \cap L^\infty$ and ζ, as well as short time continuability of the solution so long as ζ remains small. This allows us to apply the same "continuous induction" argument used in the Burgers case.

Namely, for $E_0 < 1/4C^2$, assuming nonstrict inequality $\zeta(t) \leq E_0$, we find from $\zeta(t) \leq C(E_0 + \zeta(t)^2)$ that
$\zeta(t) \leq CE_0 + C(2CE_0)^2 < 2CE_0$, yielding strict inequality. By continuity of ζ/continuability of the solution so long as ζ remains bounded, we may conclude that $\zeta \leq 2C_0E_0$ amd the solution exists for all time, with the stated bounds following by definition of ζ.

(END PROOF OF MAIN THEOREM)

Remarks

We swept a number of unneeded details under the rug (of S_H). In fact, pointwise bounds [Mascia-Z] show that the full Green kernel is a distribution involving delta-functions propagating with characteristic speed, and exponentially damped with time. For pointwise bounds in the scalar, strictly parabolic case, see the posted notes of Yingwei Liu. (More detailed analysis involving analytic stationary phase, contours inside the essential spectrum.)

NEXT TIME

In the following lecture, we will prove the stated low-frequency stability bounds (first half), and in the second half discuss the Evans function and its numerical approximation.

Blake Barker will be joining us and starting next Thursday will provide concrete details on numerical Evans computation for shock fronts and modulated periodic waves, as well as (perhaps) some discussion of his exciting progress on numerical proof.