Stability of viscous shock waves and beyond

Kevin Zumbrun

Department of Mathematics
Indiana University

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Short course, IHP: Lecture 4a

Low-frequency estimates: completion of stability proof

We have split the solution operator for linearized eqn. $U_t = LU$ into high- and low-frequency parts, $S(t)U = S_{HF}(t) + S_{LF}(t)$, where S_{HF} is time-exponentially decaying and

$$S_{LF}(t)U(x) = \int_{-\infty}^{\infty} G_{LF}(x,t;y)U(y)dy,$$

with

$$G_{LF}(x,t;y) = \frac{1}{2\pi i}\oint_{\Gamma} e^{\lambda t} G_\lambda(x,y)d\lambda, \qquad (1)$$

$G_\lambda(x,y)$ the kernel of resolvent $(\lambda - L)^{-1}$ and Γ a finite contour $\subset B(0,\varepsilon)$, $\varepsilon << 1$, with endpoints $-\eta - i\varepsilon/2$, $-\eta + i\varepsilon/2$.

GOAL

Using detailed bounds available on G_λ through our explicit representation formula, together with direct inverse Laplace transform estimates, show that

$$G_{LF}(x,t;y) = \bar{U}'(x)e(y,t) + \tilde{G}(x,t;y),$$

where $\tilde{G}$ and e_t are, up to negligible errors, given by sums of Gaussians moving with characteristic speeds $a_j^\pm \in \sigma(A_\pm$, thus justifying the $L^q \to L^p$ bounds assumed in the previous lecture and completing the proof of nonlinear stability.

I. The Evans function/construction of resolvent kernel

Recall, writing $\lambda U - LU = f$ as a first-order system $W' - \mathbf{A}(x,\lambda)W = F$ and conjugating to constant coefficient

$$Z'_{\pm} = \mathbf{A}_{\pm}(\lambda)Z = \tilde{F}_{\pm}, \qquad x \gtrless 0,$$

by transformations $W = T_{\pm}Z_{\pm}$ converging exponentially to the identity, analytic from $\lambda \to C^1(x)$, we obtain [Henry-Monteiro]:

$$\begin{aligned} Z_+(x) = e^{\mathbf{A}_+ x} P_+ Z_+(0) + \int_0 e^{\mathbf{A}_+(x-y)} P_+ \tilde{F}_+(y)dy \\ - \int_x^{\infty} e^{\mathbf{A}_+(x-y)} Q_+ \tilde{F}_+(y)dy, \end{aligned} \tag{2}$$

where P_+, Q_+ are eigenprojections onto stable, unstable subspaces of $\mathbf{A}_+$, and similarly for $x < 0$, with transmission conditions:

$$\begin{pmatrix} Q_+ T_+(0)^{-1} \\ P_- T_-(0)^{-1} \end{pmatrix} W(0) = \begin{pmatrix} -\int_0^{\infty} e^{\mathbf{A}_+(x-y)} Q_+ \tilde{F}_+(y)dy \\ \int_{-\infty}^{0} e^{\mathbf{A}_-(x-y)} P_- \tilde{F}_-(y)dy \end{pmatrix}. \tag{3}$$

Solvability

Well-posed whenever:

(i) P, Q well-defined: *consistent splitting*, $\mathbf{A}_\pm$ have spectral gaps.

(ii) $\lambda \neq$ eigenvalue, i.e., for $Q_+ = R_{Q,+}L_{Q,+}$, $P_+ = R_{P,+}L_{P,+}$, $\begin{pmatrix} L_{P,+}T_+(0)^{-1} \\ L_{Q,-}T_-(0)^{-1} \end{pmatrix}$ is invertible, or, equivalently,

$$\tilde{D}(\lambda) := \det \begin{pmatrix} L_{P,+}T_+(0)^{-1} \\ L_{Q,-}T_-(0)^{-1} \end{pmatrix} \neq 0, \tag{4}$$

yielding $Z_\pm(0) = T_\pm(0)W(0)$, with

$$W(0) = \begin{pmatrix} L_{Q,+}T_+(0)^{-1} \\ L_{P,-}T_-(0)^{-1} \end{pmatrix}^{-1} \begin{pmatrix} -L_{Q,+}\int_0^\infty e^{\mathbf{A}_+(x-y)}\tilde{F}_+(y)dy \\ L_{P,-}\int_{-\infty}^0 e^{\mathbf{A}_-(x-y)}\tilde{F}_-(y)dy \end{pmatrix}. \tag{5}$$

The Evans function

$\tilde{D}$, a locally analytic function, is one of several equivalent formulations of the *Evans function* [J. Evans, Alexander-Gardner-Jones, Pego-Weinstein, Benzoni-Serre-Z], namely, the *adjoint Evans function*.

Other versions are the *standard* (forward) Evans function

$$D(\lambda) := \det \begin{pmatrix} T_+(0)R_{Q,+} & T_-(0)R_{P,-} \end{pmatrix},$$

and the *mixed*, or Gramian, Evans function

$$\check{D}(\lambda) := \det L_{P,+} T_+(0)^{-1} T_-(0) R_{P,-}.$$

EXERCISES: 1. Using that P, Q vary analytically, show that R, L may be chosen analytically. (ii) Show that all three definitions agree up to a nonvanishing analytic multiplier.

II. Analytic extension through $\sigma_{ess}(L)$

The Evans function is (up to now) locally analytic, as, by construction, is the resolvent kernel. Noting that the resolvent kernel is *unique*, we obtain global analyticity on connected components of the domain of consistent splitting, in particular on $\{\Re\lambda \geq 0\} \setminus \{0\}$.

However, at the crucial point $\lambda = 0$ determining time-asymptotic behavior, consistent splitting is lost and the previous argument for local analyticity does not apply; *indeed,* $0 \in \partial\sigma_{ess}(L)$*.*

To obtain the desired bounds, we will need to extend the resolvent kernel up to, and in fact *beyond*, the essential spectrum boundary. (Reminiscent of scattering theory arguments.)

Expansion of neutral modes: continuation of $P_\pm$ and $Q_\pm$

Neutral modes at $\lambda = 0$: Eigenvalues μ and eigenvectors $R = (r, *)^T$ of $\mathbf{A}_\pm(\lambda)$ correspond to exponential modes $U = e^{\mu x} r$ of $0 = (\lambda - L_\pm)U = (\lambda + A_\pm \partial_x - B_\pm \partial_x^2)U$, or solutions of

$$(\lambda + \mu A_\pm - \mu^2 B_\pm)r = 0.$$

Thus, neutral modes $\mu = ik$ correspond to solutions of

$$(\lambda + ikA_\pm + k^2 B_\pm)r = 0. \tag{6}$$

At $\lambda = 0$, n-fold root $k = 0$. No others, by behavior of dispersion curves, solutions $\lambda_j(k)$ of (**??**). Moreover:

Expansion of dispersion curves

By assumed simplicity of $\sigma(A_\pm)$, and standard spectral perturbation theory, eigenvalues

$$\lambda_j(k) \in \sigma(-ikA - k^2B)_\pm = -ik\sigma(A - ikB)_\pm$$

and their left and right eigenvectors $\mathbf{l}_j(k)$, $\mathbf{r}_j(k)$ bifurcate analytically from $\lambda_j(0) = 0$, with expansions

$$\begin{aligned} \lambda_j^\pm(k) &= -ika_j^\pm - k^2\beta_j^\pm + O(k^3), \\ \mathbf{l}_j^\pm(k) &= l_j^\pm + O(k), \\ \mathbf{r}_j^\pm(k) &= r_j^\pm + O(k), \end{aligned} \tag{7}$$

$\beta_j^\pm := (l_j B r_j)_\pm$ "effective diffusion coefficients" [Kawashima].

Inversion/expansion of neutral modes

Inverting (6), get analytic expansions:

$$\begin{aligned} \mu_j^{\pm}(\lambda) &= -\lambda/a_j^{\pm} + \lambda^2\beta_j/a_j^3 + O(\lambda^3), \\ \mathbf{l}_j^{\pm}(\lambda) &= l_j^{\pm} + O(\lambda), \\ \mathbf{r}_j^{\pm}(\lambda) &= r_j^{\pm} + O(\lambda), \qquad j = 1, \ldots, n. \end{aligned} \tag{8}$$

Evidently, converting to phase (augmented) variables $Z_{\pm}$, have:

Extension of projectors

Lemma

$P_\pm$, $Q_\pm$ extend analytically in a neighborhood of $\lambda = 0$, as the sum of strong stable, unstable projectors (onto eigenmodes with nonvanishing real part), and neutral stable, unstable projectors

$$P^0_\pm = \sum_{a_j^\pm > 0}^{n} (R_j L_j^*)_\pm, \; Q^0_\pm = \sum_{a_j^\pm < 0}^{n} (R_j L_j^*)_\pm, \tag{9}$$

where $L_j^\pm = \begin{pmatrix} \mathbf{l}_j^\pm \\ * \end{pmatrix}$, $R_j^\pm = \begin{pmatrix} \mathbf{r}_j^\pm \\ * \end{pmatrix}$.

Extension of the Evans function and resolvent kernel

Corollary

The adjoint Evans function $\tilde{D}(\lambda) := \tilde{D}(\lambda) := \det \begin{pmatrix} L_{P,+} T_+(0)^{-1} \\ L_{Q,-} T_-(0)^{-1} \end{pmatrix}$ *may be extended analytically to* $\lambda \in B(0, \varepsilon)$*, for* $1 >> \varepsilon > 0$*.*

Corollary

The resolvent kernel $G_\lambda(x, y) : \lambda \to C^0(x, y)$ *extends meromorphically to* $\lambda \in \{\Re\lambda \geq 0\} \cup B(0, \varepsilon)$ *for* $\varepsilon > 0$ *sufficiently small, with poles of* G_λ *corresponding to zeros of* $\tilde{D}$*.*

Proofs. The first corollary is evident. Extension of G_λ to $B(0, \varepsilon)$ follows by definition in terms of a rational expression in the analytic functions $T_\pm$, $Q_\pm$, $P_\pm$.

(For present purposes, need only $B(0, \varepsilon)$; recall, $\Gamma \subset B(0, \varepsilon)$.)

III. The Evans condition and pointwise bounds on $G_\lambda(x, y)$

We introduce now the *Evans condition*:
(D) The (*) Evans function has a single root at $\lambda = 0$ on $\Re\lambda \geq 0$, of multiplicity $\ell = 1$ (the dimension of the family of traveling-wave connections, specialized to the Lax case $\ell = 1$).

Recalling Henry-M expansion (2), we have that $G_\lambda(x, y)$ is sum of:

1. *pole terms* $(1/\lambda)\phi(x)\psi(y)e^{-\mu_j^\pm(\lambda)y}$ (ϕ, ψ unif. bdd.),
2. *scattering terms* $\phi(x)\psi(y)e^{\mu_j^\pm(x-y)}$ and $\phi(x)\psi(y)e^{\mu_j^\pm x-\mu_k^\pm(\lambda)y}$ (ϕ, ψ unif. bdd.), and
3. *remainder terms*: of same form, but with additional factor $O(\lambda)$.

MORE: From the fact that residue at $\lambda = 0$ is projection onto zero-eigenspace of L, we find that WLOG (rescaling ψ, and collecting common terms) $\phi(x) = \bar{u}_x$ in 1. (See [Z:Handbook] for details.)

PAUSE/INTERMISSION

We're now ready to obtain the main bounds, as we will do shortly.