Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 6a

Modulation of periodic wavetrains: framework and stability

Another family of problems featuring lack of spectral gap is *stability of periodic wavetrains.*

In today's lectures, we'll study:

(a) stability, by analysis similar to that for viscous shocks.

(b) relation to hyperbolic conservation laws, through formal modulation equation

I. Periodic patterns in reaction diffusion equations

Example 1: scalar reaction diffusion equation:

$$u_t + g(u) = u_{xx}, \qquad (1)$$

e.g., $g(u) = dG(u) = (1/2)(u^2 - 1)u$, "double-well" potential:

$$G(u) = (1/8)(u^2 - 1)^2. \qquad (2)$$

Combines spatial diffusion $u_t = u_{xx}$. reaction $u_t = -dG(u)$, (Nerve impulse , pattern formation, population dynamics.)

Stationary solutions satisfy $g(\bar{u}) = \bar{u}''$, Hamiltonian system,

$$H(u) = G(u) + |u'|^2/2 = \text{constant}.$$

Family of periodic profiles filling two-cycle connecting $u = \pm 1$.

Pattern formation in systems

Example 2: Turing patterns. Homogeneous equilibrium $U \equiv U_0$, $dG(U_0) = 0$ of

$$U_t + dG(U) = DU_{xx}, \qquad U \in \mathbb{R}^n,$$

$D = \mathrm{diag}\{d_1, \ldots, d_n\} > 0$, $d^2G(U_0)$ stable, hence instabilities of linearized equation must occur at finite wavelength=Fourier frequency k, $\Rightarrow$ bifurcation to periodic (standing or traveling wave) solution.

(If supercritical, stable with respect to co-periodic perturbations.)

Details

- $dG = 0$, $d^2G > 0 \Leftrightarrow U_0$ is strict minimum of the potential ($\sim$ thermodynamic stability in the shock case).

- Fourier transform gives growth rates $e^{\lambda(k)t}$, where

$$\lambda(k) \in -(d^2G(U_0) + k^2D).$$

Hence, instabilities $\Re\lambda(k) > 0$ must occur *away from* $|k| = 0, \infty$ by stability of reaction and diffusion considered separately, thus imposing a length scale $|k|$ and spontaneouly emerging periodic patterns (the brilliant observation of Turing). This can happen in two dimensions due to noncommutation of d^2G and D, doesn't occur for scalar case...

- Turing proposed specific chemical scenario, not abstract $n = 2$...

Exercises

For $A > 0$ symmetric, $D = \mathrm{diag}\{d_1, \dots, d_n\} > 0$, consider dispersion relation

$$\lambda_j(k) \in -(d^2 G(U_0) + k^2 D). \quad j = 1, \dots, n.$$

a) for $n = 2$, show that instability $\Re\lambda \le 0$ can occur only if $d_1 \gg d_2$ or vice versa.

b) show that pure imaginary roots $\lambda(k) = \pm i\tau$, $\tau \neq 0$ real, $\sim$ traveling periodic patterns cannot occur for $n \le 2$.

c) (*) show that they *can* occur for $n \ge 3$.

Canonical model, weakly unstable (small-amplitude) case

More generally (Eckhaus, Mielke, Schneider, ...), small-amplitude waves can be described for a large variety of models by *amplitude equations*:

Complex Ginzburg Landau equation (cGL), $r = 0$:

$$A_t = (1 + i\alpha)A_{xx} + \mu A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4, \quad A \in \mathbb{C}$$

Remarks. Real GL for zero-speed waves, reflective symmetry (Rayleigh-Benard, Swift–Hohenberg).

Background $\approx$ constant-coefficient.

II. Analytic framework: Bloch transform primer

Substituting $u = \bar{u}(k(x - ct))$ into $u_t + g(u)_x = Du_{xx}$ (hence fixing period one) yields ODE

$$k^2 D\bar{u}'' = g(\bar{u}) - ck\bar{u}', \tag{3}$$

generically 1-parameter family of solutions (up to translation)

$$u(x,t) = \bar{u}^k(kx + \omega(k)t), \tag{4}$$

indexed by wavenumber k, with "nonlinear dispersion relation"

$$\omega := -ck = \omega(k).$$

Definition

Linearized equations:

$v_t = Lv := (D\partial_x^2 + c\partial_x - A)v, \quad c = \text{speed}, \quad A := dg(\bar{u}) \text{ periodic.}$

Bloch representation, $u \in L^2(R)$:

$$u(x) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} \check{u}(\xi, x) d\xi, \tag{5}$$

where $\check{u}(\xi, x) \in L^2[0,1]_{periodic}$ is **Bloch transform**

$$\check{u}(\xi, x) := \sum_{j \in Z} \hat{u}(\xi + 2\pi j) e^{2\pi i j x} = \sum_{j \in Z} u(x + j) e^{-i\xi(x+j)}. \tag{6}$$

(Equivalent to Poisson summation formula, $x, \xi = 0$, $\sim$ Dirac comb.) For references, theory, see, e.g., thesis of Bernhard Barth: digbib.ubka.uni-karlsruhe.de/volltexte/documents/2916825

Properties

Bloch transform $u \to \check{u}$,

$$\check{u}(\xi, x) := \sum_{j \in Z} \hat{u}(\xi + 2\pi j) e^{2\pi i j x} = \sum_{j \in Z} u(x + j) e^{-i\xi(x+j)}.$$

- L^2 (H^s) isometry.

(Isometry of discrete and continuous Fourier transforms + Fubini for Schwartz class functions, extension by density/continuity.)

- Hausdorff-Young inequalities * $(q \leq 2 \leq p,\ \frac{1}{p} + \frac{1}{q} = 1)$:

$$|u|_{L^p(x)} \leq |\check{u}|_{L^q(\xi, L^p([0,X])} \quad \text{and} \quad |\check{u}|_{L^p(\xi, L^q([0,X])} \leq |u|_{L^q(x)}. \qquad (7)$$

($q = 2$ by L^2 isometry and at $q = 1$ by triangle inequality, using alternative representations above; $1 < q < 2$ by analytic interpolation argument, as standard H-Y.)

Floquet decomposition

It follows that the L^2 spectra of L are given by

$$\sigma(L) = \cup_{\xi \in \mathbb{R}} \sigma(L_\xi), \tag{8}$$

where

$$L_\xi := e^{-i\xi x} L e^{i\xi x} = (\partial_x + i\xi)^2 - (\partial_x + i\xi)A.$$

Proof: L^2 isometry and $Lu = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} e^{i\xi x} L_\xi \check{u}_0(\xi, x) d\xi$ by the identity $Le^{i\xi x} = e^{i\xi x} L_\xi$.)

Alternative proof: Floquet's Lemma applied to the resolvent equation reduces to constant coeffients, whence (8) reduces (for each individual λ) to Fourier decomposition. (Or, direct reasoning based on monodromy map [Gardner, ...].)

Spectral decomposition formula: inverse Bloch transform representation of the solution

The solution operator for linearized evolution equation $u_t = Lu$ on $L^2(\mathbb{R})$ may be expressed as

$$\begin{aligned} S(t)u_0 &= \Big(\frac{1}{2\pi}\Big)\int_{-\pi}^{\pi} e^{i\xi x} S_\xi(t)\check{u}_0(\xi,x)d\xi \\ &=*\Big(\frac{1}{2\pi}\Big)\int_{-\pi}^{\pi} e^{i\xi x}\sum_j e^{\lambda_j(\xi)t} q_j(\xi,x)\langle\tilde{q}_j(\xi,\cdot,\rangle d\xi, \end{aligned} \tag{9}$$

where λ_j, q_j, $\tilde{q}_j$ are e-values, right/left e-fns of

$$L_\xi := e^{-i\xi x} L e^{i\xi x} = (\partial_x + i\xi)^2 - (\partial_x + i\xi)A,$$

and $S_\xi(t)$ is the solution operator for $u_t = L_\xi u$ on $L^2([0,1]_{per})$.

(Formally, $S_\xi(t) = e^{-i\xi x} S(t) e^{i\xi x}$; (*) if e-values semisimple.)

Details

• Both $S(t)$ and $S_\xi(t)$ may be constructed by the inverse Laplace transform approach used previously for the shock wave case, together with resolvent bounds obtained using energy estimates and direct computation by Floquet's lemma applied to the resolvent equation. Both are unique (by energy estimates, for classical strong solutions, extension by continuity/density).

• Applying ∂_t to both sides, result follows for classical solutions by identity $(\check{Lu}) = L_\xi \check{u}$; extension by continuity/density.

Remarks

1. As for the Laplace transform, we have obtained Bloch tranform facts by relating to Fourier transform.

2. **DIAGONALIZES PERIODIC-COEFFICIENT OPERATORS:** Bloch is natural analog for periodic-coefficient operators of the Fourier transform in the constant-coefficient case.

III. Consequences for periodic wave trains

We now observe

TWO SIMPLE CONSEQUENCES:

(i) Lack of spectral gap!

(ii) (compensating for (i)) Diffusive $L^q \to L^p$ bounds...

Consequence (i): lack of spectral gap for traveling waves

Differentiating traveling-wave ODE $k^2 D\bar{u}'' = g(\bar{u}) - ck\bar{u}'$ with respect to x yields

$$0 = L\bar{U}_x = (k^2 D\partial_x^2 - dg(\bar{U}) - ck\partial_x)\bar{U}'.$$

Thus, $\lambda = 0$ is always an eigenvalue of L_0 (=L restricted to periodic functions), hence:

for periodic waves, there exists a curve of "neutral" *spectra* $\lambda_*(\xi)$ *tangent to the imaginary axis at* "translational e-value" $\lambda = 0$*.*

Consequence (ii): Diffusive $L^1 \to L^p$ bounds, $p \geq 2$

Heat equation $\widehat{e^{\Delta t} f} = e^{-\xi^2 t} \hat{f}$:

$$\begin{aligned} \|\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x} e^{-\xi^2 t}\hat{f}(\xi) d\xi\|_{L^p(x)} &\leq \|e^{-\xi^2 t}\|_{L^q(\xi}\|\hat{f}\|_{L^\infty(\xi)} \\ &\leq t^{-\frac{1}{2}(1-1/p)}\|f\|_{L^1(x)}. \end{aligned} \tag{10}$$

Simple periodic eigenmode, "diffusive" dependence $\Re\lambda(\xi) \leq -\xi^2$:

$$\begin{aligned} &\|\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\xi x} e^{\lambda(\xi) t} q(\xi, x)\langle \tilde{q}, \check{f}\rangle(\xi) d\xi\|_{L^p(x)} \\ &\qquad \leq \|e^{-\xi^2 t}\|_{L^q(\xi}\|\|q\|_{L^\infty(\xi, L^p(x))}\|\tilde{q}\|_{L^\infty(\xi,x)}\|\check{f}\|_{L^\infty(\xi, L^1(x))} \\ &\qquad \leq t^{-\frac{1}{2}(1-1/p)}\|f\|_{L^1(x)}. \end{aligned} \tag{11}$$

(Plus general $L^q \to L^p$ bounds, $1 \leq 2 \leq p$)

Similarly,

$$\|\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\xi x} e^{\lambda(\xi)t} q(\xi,x)\langle \tilde q, \check f\rangle(\xi)d\xi\|_{L^p(x)}$$
$$\leq \|e^{-\xi^2 t}\|_{L^{2q/(2-q)}(\xi}\|\|q\|_{L^\infty(\xi,L^p(x))}\|\tilde q\|_{L^\infty(\xi,x)}\|\check f\|_{L^2(\xi,L^2(x))}$$
$$\leq t^{-\frac{1}{2}(1/2-1/p)}\|f\|_{L^2(x)}, \tag{12}$$

and, by interpolation,

$$\|\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\xi x} e^{\lambda(\xi)t} q(\xi,x)\langle \tilde q, \check f\rangle(\xi)d\xi\|_{L^p(x)} \leq t^{-\frac{1}{2}(1/q-1/p)}\|f\|_{L^2(x)}.$$

- **SAME AS FOR HEAT EQUATION** -

IV. Nonlinear stability

Schneider (1995) showed nonlinear stability of of *zero-speed* periodic waves under localized perturbations by a brilliant nonlinear cancellation/renormalization group argument, answering a then 30-year-old open question.

PICTURE: phase-modulation by perturbation (phase-shift) $\psi(x, t)$ satisfying

$$\psi_t = \delta(\psi_x)_x,$$

(motivated by formal Ginzburg-Landau modulation equations,) hence decaying as a Gaussian for initially localized perturbations.

We will recover this result for arbitrary-speed waves by a simple argument generalizing our shock stability analysis [Johnson-Z10].

Diffusive stability conditions [Schneider95]

(D1) $\sigma(L_\xi) \subset \{\mathrm{Re}\lambda < 0\}$ for $\xi \neq 0$.
(D2) $\mathrm{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}$ and $|\xi|$ sufficiently small.
(D3) $\lambda = 0$ is a simple eigenvalue of L_0 (transversality in traveling wave ODE).

(D1)–(D3) $\Rightarrow$ analytic eigenvalue

$$\lambda_*(\xi) = -ia_j\xi + O(|\xi|) \tag{13}$$

of L_ξ bifurcating from $\lambda = 0$, with associated analytic eigenprojector

$$\Pi(\xi) = q(\xi, \cdot)\langle\tilde{q}(\xi, \cdot), \cdot\rangle = \Pi(0) + O(|\xi|), \tag{14}$$

q and $\tilde{q}$ (analytic) right and left eigenfunctions.

Basic nonlinear stability result

Proposition (Johnson-Z10)

Let $\tilde u$ be a solution of $u_t + g(u) = Du_{xx}$. Assuming smoothness of g, (D1)–(D3), and $E_0 := \|\tilde u - \bar u\|_{L^1\cap H^4}|_{t=0}$ sufficiently small,

$$\begin{aligned}
\|\tilde u - \bar u(\cdot - \psi)\|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}E_0,\\
\|(\psi_t,\psi_x)\|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}E_0,\\
\|\psi(t)\|_{L^p},\ \|\tilde u - \bar u(\cdot)\|_{L^p}(t) &\le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0,
\end{aligned}$$

for some $\psi(x,t)$, $C>0$, and all $t\ge 0$, $p \ge 2$.

Comments

1. From the proposition, asymptotic behavior of perturbation is

$$\tilde{u}(x,t) \approx \bar{u}(x+\psi(x,t)) \approx \bar{u}(x) + \bar{u}_x \psi(x,t), \tag{15}$$

in agreement with (linearization of) formal modulation equations.

2. Schneider in fact shows $\tilde{u} \sim \bar{u} + \bar{u}_x A e^{-x^2/ct}/\sqrt{t}$, obtaining behavior as well as decay. We'll show in the second lecture more recent results on behavior from a different point of view, that include and substantially extend this.

(**END FIRST LECTURE**)