Stability of viscous shock waves and beyond

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Sponsored by NSF Grants no. DMS-0300487 and DMS-0801745

Short course, IHP: Lecture 15a

Reprise

In this lecture, RETURN TO ROOTS (= shock waves):

• Original motivations for study: what progress have we made?

• Thermodynamical assumptions: Bethe-Weyl conditions and the role of entropy in viscous and inviscid shock stability.

• Numerical proof: toward viscous Majda's Theorem for γ-law gas.

I. A 25-year summer research project: inviscid instability, shock splitting, and the viscous selection principle

POSED BY BURTON WENDROFF (Los Alamos internship, summer 1990):

Interesting examples of Smith (more later) of "real gas" satisfying usual Bethe-Weyl conditions but exhibiting nonuniqueness of Riemann solutions. Numerics indicate physical selection principle via *stability of component shocks*. But, not inviscid stability (in fact, both inviscid-stable); rather, numerical viscosity...

QUESTION: Can we validate/confirm this observation by rigorous by rigorous stability analysis?

Interest redoubled in postdoc investigations: undercompressive shocks in multi-phase flow (Glimm, Marchesin, Plohr).

Revisiting an observation of C. Gardner: the signed Lopatinski condition and uniqueness of Riemann solutions

Post-world war II years: D'yakov, Kondrachov, Erpenbeck derived inviscid stability (Lopatinski) condition. Relation of 1D condition to uniqueness was pointed out by C. Garner for gas dynamics, giving simple sufficient condition for stability (monotonicity of Hugoniot curves).

For a general system of conservation laws

$$u_t + f(u)_x = 0,$$

Rankine-Hugoniot (shock) conditions:

$$\sigma[u] = [f], \tag{RH}$$

yielding free-boundary/transmission problem, n constraints, unknowns consisting of speed σ and outgoing modes.

The Lopatinski condition

Counting variables yields *Lax condition*, number of outgoing modes $= n - 1$. Checking rank yields *Lopatinski* (determinant) *condition*

$$\delta := \det(r_1^-, \dots, r_{p-1}^-, [u], r_{p+1}^+, \dots, r_n^+) \neq 0, \qquad (1)$$

where r_j^-, $j = 1, \dots, p-1$ are the "outgoing" modes of $A_- := f_u(U_-)$ associated with eigenvalues $a_j^- - \sigma < 0$ and r_j^+, $j = p+1, \dots, n$ are the "outgoing" modes of $A_+ := f_u(U_+)$, associated with eigenvalues $a_j^+ - \sigma > 0$.

(Multi-D analog by Laplace-Fourier analysis, following approach discussed for hyperbolic IVBP in lectures of Métivier.)

Relation to Riemann solutions

Riemann ("shock tube") problem: self-similar solution $u(x/t)$ for discontinuous data $u_0 = u_-$ for $x < 0$ and u_+ for $x \geq 0$. Constructed from Riemann (=rarefaction + shock) curves in characteristic families $j = 1, \dots, n$, as

$$u_+ = \Psi_n^{\theta_n} \cdot \Psi_1^{\theta_1}(u_-),$$

where θ_j parametrize motion along Riemann curves. Fixing u_-, uniqueness amounts to invertibility of the $\mathbb{R}^n \to \mathbb{R}^n$ map

$$\mathcal{R} : (\theta_1, \dots, \theta_j) \to u_+.$$

At a viscous p-shock, $\theta = (0, \dots, 0, \theta_p, 0, \dots 0)$, Jacobian of $\mathcal{R} \sim$ Lopatinski determinant $\delta = \det(r_1^-, \dots, r_{p-1}^-, [u], r_{p+1}^+, \dots, r_n^+)$!

Shock splitting: relation to nonuniqueness

Now, fix u_- and move along the pth Hugoniot curve

$$\theta = (0, \cdots, 0, \theta_p, 0, \dots, 0).$$

Proposition ([Barker-Freistühler-Z14)

Change in sign of δ as θ_p crosses θ_p^* implies nonuniquess of Riemann solutions in the vicinity of the shock solution (u_-, u_+^*) associated with θ_p^*.

Proof.

Uniqueness would imply constancy of Brouwer degree of $\mathcal{R}$ in a vicinity of $\theta_* = (0, \cdots, 0, \theta_p^*, 0, \dots, 0)$ while at the same time equality of the degree to the sign of the Jacobian of $\mathcal{R}$. □

Result: double-valued solution with alternating-sign shocks

Generically, corresponds to fold singularity, double (resp. zero) valued solution on one side of image of the singularity curve.

Moreover, preimages (since Brouwer degree zero is invariant) have Jacobians of opposite sign (summing to zero).

(Simple example is 2×2 case [Z01].)

Now recall Evans function theory

For small θ, associated shock profiles exist/are transversal connections of $u_\pm$ in general [Majda-Pego, Freistühler]. Assuming that these properties continue to large θ, as holds for example for a Weyl-Bethe gas [Gilbarg51, Menikoff-Plohr89], we may consider the related question of *viscous stability.*

In particular, associated to the linearized problem about a viscous profile, the is an *Evans function* $D(\lambda)$, analytic on $\Re\lambda \geq 0$, and satisfying the low-frequency relation

$$D'(0) = \gamma\delta,$$

where γ is a (nonvanishing by assumption) Wronskian associated with the linearized traveling-wave ODE. Moreover, D may be taken real-valued for γ real.

Conclusion

Evidently, the *Stability index*

$$\mathrm{sgn} D'(0) D(+\infty_{real})$$

counts parity of the number of nonzero nonstable roots $\Re\lambda \geq 0$.

Hence, change of sign in δ is generically associated with passage of a root of the Evans function from the unstable ($\Re\lambda < 0$) to the stable ($\Re\lambda > 0$) complex half plane.

Corollary: One of the two Riemann solutions features a viscous unstable shock (the one with wrong sign of δ). Note that neither is inviscid unstable since $\delta \neq 0$. In the case that this passage of a root through $\lambda = 0$ is the *first* crossing for the viscous problem (typically the case), then the other shock is stable and the associated Riemann solution is indeed selected by the viscous selection principle.

Comments

• Rather strong conclusions drawn from topological considerations, essentially no computation. No counterpart in inviscid theory.

• Stability index is related to pioneering work of Pego-Weinstein in dispersive scalar case. In some cases, directly computable, yielding instability/stability results [Gardner-Z, homoclinic uc case; Oh-Z, periodic case]. Surprisingly, noncharacteristic viscous boundary layers are always *unstable* for γ-law gase, in the large-amplitude standing-shock limit [Serre-Z, Z].

• The corresponding low-frequency relation for multi-D yields the perhaps surprising result that viscosity can only *destabilize* and not stabilize an inviscid shock wave [Z-Serre99].

II. Uniqueness, entropy, and viscous/inviscid shock stability

(with Barker, Freistühler, Texier)

Quasilinear hyperbolic–parabolic conservation laws:

$$u_t + f(u)_x = \begin{cases} 0 \\ \nu(b(u)u_x)_x \end{cases}, \quad u \in R^n,\ \nu > 0,$$

govern compressible (gas, plasma, solid) mechanics. Interesting solutions [Riemann1856] [Rayleigh1919,Gilbarg1951] are shock waves

$$u(x,t) = \bar{u}\Big(\frac{x - \sigma t}{\nu}\Big), \qquad \lim_{x \to \pm\infty} u(x) = u_\pm,$$

propagating large energies coherently over great distance,
Stationary solutions of $u_t - \sigma u_x + f(u)_x = \nu(b(u)u_x)_x$.

Stability Theory

Studied since 1960's (1980's) [Erpenbeck, Landau, Dy'akov,...] [Oleinik, Matsumura, Nishihara, Kawashima, Goodman, Liu, ...] In principle, well understood.

Stability criteria: reduce to spectra of linearized equations, computable as zeros of a *Lopatinski* (*Evans*) determinant [Majda1983, Métivier1992,...][Gardner,Howard, Mascia, Serre, Zumbrun, 1998-2006].

All-parameters stability analyses: Combining asympototic analysis/singular perturbations and numerical Evans analysis, determine stability across full parameter range [Barker, Humpherys, Lafitte, Lewicka, Lyng, Zumbrun, 2009-2012].

Stability Practice

Essentially all waves studied so far (gas, MHD, viscoelasticity, with polytropic gas laws) are stable!

Could there be a simple mechanism/structure for this?

In particular, could *thermodynamic stability*, or convexity of the equation of state $e = \bar{e}(\tau, S)$, imply shock stability?

More generally, *existence of convex entropy*? For $u_t + f(u)_x = \nu(bu_x)_x$,

$$\eta(u), \qquad d\eta df = dq, \qquad d^2\eta(u)b \geq 0, \qquad \partial_t \int \eta(u) \leq 0.$$

(Associated with symmetrizability, stability of constant solutions.)

Further questions

It is known [Gardner-Z1998,Z-Serre1999] that viscous stability implies inviscid stability. **Is this strict? Or do they coincide?**

"Viscous destabilization" must occur through passage of nonzero imaginary roots (same low-frequency analysis), so amounts to *Hopf bifurcation*, or "galloping," familiar in detonation.

Could convex entropy prevent existence of nonzero imaginary eigenvalues for the viscous problem? Or, give a "principle eigenvalue property" preventing imaginary leading mode?

(In either case, viscous and inviscid stability would coincide.)

III. Results: Inviscid gas dynamics

Our inviscid analysis is related to/partly based on important work of R. Smith [Smith1979] on uniqueness of Riemann solutions.

Equations. In Lagrangian coordinates, the Euler equations are

$$\begin{aligned} \tau_t - v_x &= 0, \\ v_t + p_x &= 0, \\ (e + v^2/2)_t + (vp)_x &= 0, \end{aligned} \tag{2}$$

where τ denotes specific volume, v velocity, e specific internal energy, and p pressure. Here, $p = \hat{p}(\tau, e)$ is obtained by inverting $S = \hat{S}(\tau, e)$, using $T = \bar{e}_S > 0$ and $p = -\bar{e}_\tau$.

Assumptions

Ideal gas assumptions [Weyl, Bethe]:

$$p = \hat{p}(\tau, e) > 0. \quad \text{(Positivity)} \tag{J1}$$

$$(\partial_\tau - p\partial_e)\hat{p} < 0. \quad \text{(Hyperbolicity)} \tag{J2}$$

$$(\partial_\tau - p\partial_e)^2\hat{p} > 0. \quad \text{(Genuine nonlinearity)} \tag{J3}$$

$$\hat{p}_e > 0. \quad \text{(Weyl condition)} \tag{J4}$$

Conditions

Stability/uniqueness conditions [Smith, BFZ]:

$$-\frac{\bar{e}_{\tau s}}{\bar{e}_S \bar{e}_{\tau\tau}} < -\frac{1}{\bar{e}_\tau}, \text{ or } \hat{p}_\tau < 0, \qquad \text{(Strong)}$$

$$-\frac{\bar{e}_{\tau} S}{\bar{e}_S \bar{e}_{\tau\tau}} < -\frac{\frac{\bar{e}_\tau^2}{2e\bar{e}_{\tau\tau}} + 1}{\bar{e}_\tau}, \text{ or } \hat{p}_\tau < \frac{p^2}{2e}, \qquad (\text{Medium}_{\mathrm{U}})$$

$$-\frac{\bar{e}_{\tau} S}{\bar{e}_S \bar{e}_{\tau\tau}} < -\frac{-\frac{\bar{e}_\tau}{\sqrt{2e\bar{e}_{\tau\tau}}} + 1}{\bar{e}_\tau}, \text{ or } \hat{p}_\tau < cp/\sqrt{2e}, \qquad (\text{Medium}_{\mathrm{S}})$$

$$-\frac{\bar{e}_{\tau s}}{\bar{e}_S \bar{e}_{\tau\tau}} < -\frac{2}{\bar{e}_\tau}, \text{ or } \hat{p}_\tau < \frac{p\hat{p}_e}{2}. \qquad \text{(Weak)}$$

Main inviscid results

Theorem

Assuming (J1)–(J4), *stability (for all shocks) is equivalent to* (Medium$_{\mathrm{S}}$) *while [Smith] uniqueness of Riemann solutions (for any data) is equivalent to* (Medium$_{\mathrm{U}}$). *The four conditions are related by*

$$\text{(Strong)} \Rightarrow (\text{Medium}_{\mathrm{U}}) \Rightarrow (\text{Medium}_{\mathrm{S}}) \Rightarrow \text{(Weak)}.$$

In particular, condition (Strong) *by itself is sufficient to imply stability of all shocks, while violation of* (Weak) *implies existence of unstable ones.*

Corollary

The equation of state $\bar{e}(\tau, S) = \frac{e^S}{\tau} + C^2 e^{S/C^2 - \tau/C}$, $C >> 1$, *is convex, satisfies* (J1)–(J4), *but admits (inviscid) unstable shocks.*

Viscous results (i) (numerical)

Numerical Observation A (on gas dynamics). For the above equation of state, (a) viscous [in]stability is equivalent to inviscid [in]stability,
(b) the viscous-stability problem has no non-zero imaginary eigenvalues; in particular, passing through the origin, and
(c) in situations of instability, the eigenvalue with largest real part is real and simple.

Viscous results (ii) (numerical)

Numerical Observation B (on general systems). There exist 3×3 viscous systems of conservation laws with convex entropy, that
(a) admit shocks that are inviscidly stable, but viscously unstable,
(b) the viscous-stability problem sometimes does have non-zero imaginary eigenvalues, while
(c) in all situations of instability we investigated numerically, the eigenvalue with largest real part is real, and transitions from stability to instability occur exclusively by real eigenvalues passing through the origin,
(d) in some cases, there are an even number of unstable (and all real) eigenvalues, and
(e) in some cases the eigenvalue with largest real part is not simple.

Discussion

• [C. Gardner1963] as a footnote states that instability is possible for convex entropy, but with incorrect example $\bar{e}(\tau, S) = e^S/\tau + f(S)$, $f' >> 1$ (satisfies strong, so stable).

• Relative entropy, [Diperna1979,Leger-Vasseur2011]; see also [Golse-Saint Raymond2004]. A question of current interest!

IV. Inviscid analysis: Hugoniot, Lax, and Lopatinski

Integrating TW-ODE $-\sigma u' + f(u)' = 0$ gives jump conditions (Hugoniot relations)

$$\sigma[u] = [f], \qquad \text{(RH)}$$

yielding free-boundary/transmission problem, n constraints, unknowns consisting of speed σ and outgoing modes.

Counting variables yields *Lax condition*, number of outgoing modes $= n - 1$. Checking rank yields *Lopatinski* (determinant) *condition*

$$\delta := \det(r_1^-, \dots, r_{p-1}^-, [u], r_{p+1}^+, \dots, r_n^+) \neq 0, \qquad (3)$$

where r_j^-, $j = 1, \dots, p-1$ are the "outgoing" modes of $A_- := f_u(U_-)$ associated with eigenvalues $a_j^- - \sigma < 0$ and r_j^+, $j = p+1, \dots, n$ are the "outgoing" modes of $A_+ := f_u(U_+)$, associated with eigenvalues $a_j^+ - \sigma > 0$.

Specialization to gas dynamics

1-*Hugoniot curve*: Combining the three equations (RH), we obtain $[e] + (1/2)(p + p_-)[\tau] = 0$, with $p = \hat{p}(\tau, e)$ determining a curve in τ, e.

Lax condition: Characteristics 0, $\pm c$, 0, $\Rightarrow$ Lax 1-shock condition $|\sigma| < c_+$, where $c = \sqrt{\bar{e}_{\tau\tau}}$ = sound speed. (Here, $\sigma < 0$.)

(Signed) Lopatinski condition: Straightforward (*) computation gives $-\left(\frac{\bar{e}_{\tau S}}{\bar{e}_S \bar{e}_{\tau\tau}}\right)_+ < \frac{\frac{|\sigma|}{c_+}+1}{[p]}$.

Monotonicity condition: IFT gives $-\left(\frac{\bar{e}_{\tau S}}{\bar{e}_S \bar{e}_{\tau\tau}}\right)_+ < \frac{\frac{\sigma^2}{c_+^2}+1}{[p]}$.

Simplified global conditions

Define now

$$-\left(\frac{\bar{e}_{\tau S}}{\bar{e}_S \bar{e}_{\tau\tau}}\right)_+ < \frac{1}{[p]}. \qquad \text{(Strong')}$$

$$-\left(\frac{\bar{e}_{\tau S}}{\bar{e}_S \bar{e}_{\tau\tau}}\right)_+ < \frac{2}{[p]}. \qquad \text{(Weak')}$$

Using $0 < |\sigma|/c_+ < 1$, we have the string of implications

$$\text{(Strong')} \Rightarrow \text{monotone} \Rightarrow \text{Lopatinski} \Rightarrow \text{(Weak')}. \qquad (4)$$

Proof of Main Theorem

Proof.

Under (J1)–(J4) we have [Weyl, Bethe], denoting as the backward 1-Hugoniot through U_+, $H_1'(U_+)$, the set of all left states U_- connected to U_+ by a Lax 1-shock,

$$[\tau] < 0 \text{ on } H_1'(U_+), \tag{P1}$$

$$p \to 0 \text{ as } U \text{ progresses along } H_1'(U_+), \tag{P2}$$

$$e \to 0 \text{ as } U \text{ progresses along } H_1'(U_+), \tag{P3}$$

$$\tau \text{ is increasing and } p \text{ decreasing along } H_1'(U_+). \tag{P4}$$

(Recall, $[e] + (1/2)(p + p_-)[\tau] = 0$.) Thus, worst case is $(e_-, p_-) = (0, 0)$, $\tau_- = 2e_+/p_+ + \tau_+$, giving result. □

Proof of Corollary (example of instability)

Proof.

$$T = \bar{e}_S = e^S/\tau + e^{S/C^2 - \tau/C} > 0,$$

$$p = -\bar{e}_\tau = e^S/\tau^2 + Ce^{S/C^2 - \tau/C} > 0,$$

$$p_S = -\bar{e}_{S\tau} = e^S/\tau^2 + C^{-1}e^{S/C^2 - \tau/C} < 0,$$

$$p_\tau = -\bar{e}_{\tau\tau} = -2e^S/\tau^3 - e^{S/C^2 - \tau/C} < 0,$$

$$p_{\tau\tau} = -\bar{e}_{\tau\tau\tau} = 6e^S/\tau^4 + C^{-1}e^{S/C^2 - \tau/C} > 0,$$

At $\tau_+, S_+ = 1, 0$, by $C >> 1$, we have failure of (Weak):

$$\frac{-\bar{e}_{S\tau}}{\bar{e}_{\tau\tau}} - \frac{2\bar{e}_S}{-e_\tau} = \frac{1 + O(C^{-1})}{3 + O(C^{-1})} - \frac{2 + O(C^{-1})}{C + O(1)} \sim \frac{1}{3} > 0.$$

□

Numerics: Hugoniot and transition point

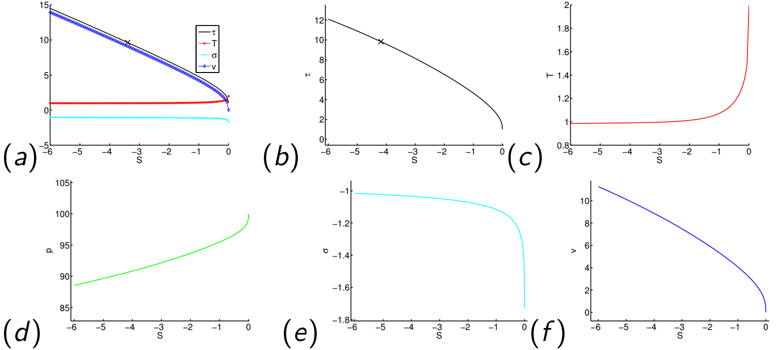

Figure: The backward 1-Hugoniot curve through $(\tau_+, S_+) = (1, 0)$ for global model $\bar{e}(\tau, S) = e^S/\tau + e^{S/C^2 - \tau/C}$ of points (τ, S) connecting to (τ_+, S_+) by a Lax 1-shock, displayed as a graph (τ, p, v, e, σ) over S plotted with respective colors (black, green, blue, red, cyan). We zoom in to see (b) the Hugoniot curve, (c) T over S, (d) p over S, (e) σ over S, and (f) v over S. The value of (τ_-, S_-) along the backward Hugoniot curve at which transition to instability occurs is marked by a black X.

Canonical local model

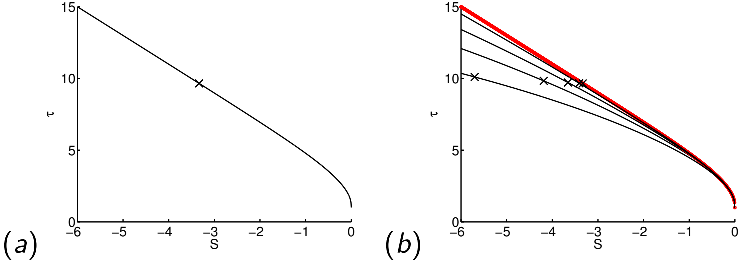

Figure: We mark transition to instability with a thick X. (a) Plot of the Hugoniot curve for the local model, $\bar{e}(S,\tau) = e^S/\tau + S + \tau^2/2$, $(\tau_0, S_0) = (1, 0)$. (b) Plot of the the Hugoniot curve for the local model with a thick red line and that of the global model for $C = 40,\ 100,\ 250$. Note that as $C \to \infty$, the Hugoniot curve of the local model matches well that of the global model.

Pause for second lecture

BREAK