Stability of viscous shock waves and beyond

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Short course, IHP: Lecture 2b

Asymptotically constant-coefficient operators and ILT representation of the Green distribution

$L = \partial_x (B(x)\partial_x + \partial_x A(x)$, limits $A_\pm$, $B_\pm$ as $x \to \pm\infty$.

GOAL: Express Solution operator $S(t)$ for $u_t = Lu$ as

$$u(x,t) = \int_{\mathbb{R}} G(x,t;y)u(y)dy, \tag{1}$$

(distributional sense), via **ILT formula**:

$$G(x,y) = P.V. \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} G_\lambda(x,y) d\lambda, \tag{ILT}$$

where $G_\lambda(x,y)$ is resolvent kernel associated with $(\lambda - L)^{-1}$.

(Formally, (ILT) applied to $\delta_y(x)$...)

Construction of the resolvent kernel: the conjugation lemma

By introducing phase variables, we can reduce resolvent equation $(\lambda - L)u = f$ to first-order equation

$$W' - A(\lambda, x)W = F. \tag{2}$$

Example: $L = \partial_x^2 + A\partial_x$, $W := (u, u')^t$, $A = \begin{pmatrix} 0 & \mathrm{Id} \\ \lambda & A \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$.

Reduction to constant coefficients

The **conjugation lemma** [Métivier-Z], analog for asymptotically constant ODE to Floquet's Lemma in the periodic case.

Lemma

For $A(\lambda)$ converging exponentially to $A_\pm(\lambda)$, there exist coordinate changes $T_\pm(\lambda)$ on $x \gtrless 0$, converging exponentially to Id as $x \to \pm\infty$, such that $W = T_\pm Z_\pm$, $F = T_\pm \tilde F_\pm$ reduces resolvent eq. (2) to constant coefficients:

$$Z_\pm' - A_\pm(\lambda) Z_\pm = \tilde F_\pm.$$

Moreover, $T_\pm$ retains the regularity in λ of A (in this case analytic).

Proof (postponed): Contraction mapping/Lyapunov-Perron [Levinson] plus homological equations for conjugation of A to $A_\pm$.

Construction of Henry (Monteiro)

Lyapunov-Perron: Projecting const.-coeff. equation onto eigencomponents, write:

$$Z_+(x) = e^{A_+ x} P_+ Z_+(0) + \int_0^x e^{A_+(x-y)} P_+ \tilde{F}_+(y) dy - \int_x^\infty e^{A_+(x-y)} Q_+ \tilde{F}_+(y) \quad (3)$$

where P_+, Q_+ are eigenprojections onto stable, unstable subspaces of A_+, and similarly for $x < 0$; matching conditions:

$$T - Z_-(0) = T_+ Z_+(0).$$

ASSUMPTIONS (i) (ODE) *hyperbolicity*, $\sigma(A)$ has nonvanishing real part. (ii) *consistent splitting*, dimensions of stable/unstable subspaces same for $A_\pm$.

Remark. Domain of consistent splitting bounded by dispersion curves of $A_\pm$, spectra of limiting constant-coeff. operators $L_\pm$.

Solving for kernel

Obs. Existence of eigenvalue equivalent to solution with $\tilde{F} = 0$, i.e., nontrivial $Z(0)$ s.t. $Q_+ Z+(0)$, $Q_- Z-(0) = 0$.

Setting $x = 0$ in (3), get

$$Q_+ Z_+(0) = \int_0^\infty - \int_x^\infty e^{A_+(-y)} Q_+ \tilde{F}_+(y) dy,$$

and similarly for $x < 0$. If no e-value, then this determines $Z_\pm(0)$, by matching conditions, bounded linear operation.

COMBINING, we have $W(x) = \int \mathcal{G}_\lambda(x, y) F(y) dy$, where

$$|G_\lambda(x, y)| \leq C e^{-\eta|x-y|},$$

exponential decay in $|x - y|$.

Remark. Uniqueness clear from construction (=solution).

Henry's Theorem

From convolution bound $|f * g|_p \leq |f|_p |g|_1$, and evident L^1 bound on our bound $Ce^{-\eta|x-y|}$ for $|G_\lambda(x,y)|$, get L^p boundedness of $(\lambda - L)^{-1}$ on set of consistent splitting, whever λ is not an eigenvalue.

Corollary (Henry)

$\sigma_{ess}(L)$ lies to the left of the rightmost dispersion curve of the limiting operators $L_\pm$ (equivalently, their curves of essential spectra).

BUT: for nonself-adjoint operators, can be open regions of essential spectrum where consistent splitting fails (easy examples). Thus, ILT is different from generalized Fourier decomposition...

High-frequency bounds

Previous bounds *not uniform in* λ. For high-frequency (large λ) bounds, use different, semiclassical limit-type, asymptotic ODE estimates (e.g., "tracking," or "reduction" lemmas of [Z-Howard, Handbook], etc.)

IDEA: For rapidly varying solutions (large λ), behaves approximately as "frozen coefficient equation," tracks close to stable/unstable subspaces. We will omit these (for now), as we won't actually use them in our stability analysis.

Result: Uniform bounds $|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}$, ind. of λ, $\Re\lambda$ sufficiently large.

II. Verification of ILT formula

Validity of ILT for Green distribution now follows by splitting the operator ILT integral as before, then observing that all terms are of form L^k applied to an absolutely convergent integral in λ and $L^p(\mathbb{R})$, hence converges to a distribution$=$ L^k applied to L^2 function.

NEXT

We will use the more detailed Green distribution description in order to obtaion our desired linearized estimates.

NOTE: Again, similarly as in operator-ILT, that the representation formula is only a starting point. Once we obtain bounds on the solution (which, for distribution, means on veryregular test function data), these *extend to the full semigroup*.