PERIODIC-COEFFICIENT DAMPING ESTIMATES, AND STABILITY OF LARGE-AMPLITUDE ROLL WAVES IN INCLINED THIN FILM FLOW

L.MIGUEL RODRIGUES AND KEVIN ZUMBRUN

Abstract. A technical obstruction preventing the conclusion of nonlinear stability of large-Froude number roll waves of the St. Venant equations for inclined thin film flow is the "slope condition" of Johnson-Noble-Zumbrun, used to obtain pointwise symmetrizability of the linearized equations and thereby high-frequency resolvent bounds and a crucial H^s nonlinear damping estimate. Numerically, this condition is seen to hold for Froude numbers $2 < F \lessapprox 3.5$, but to fail for $3.5 \lessapprox F$. As hydraulic engineering applications typically involve Froude number $3 \lessapprox F \lessapprox 5$, this issue is indeed relevant to practical considerations. Here, we show that the pointwise slope condition can be replaced by an averaged version which holds always, thereby completing the nonlinear theory in the large-F case. The analysis has potentially larger interest as an extension to the periodic case of a type of weighted "Kawashima-type" damping estimate introduced in the asymptotically-constant coefficient case for the study of stability of large-amplitude viscous shock waves.

1. Introduction

The St. Venant equations of inclined thin film flow, in nondimensional Lagrangian form, are

$$\begin{aligned} &\partial_t \tau - \partial_x u = 0, \\ &\partial_t u + \partial_x \left(\frac{\tau^{-2}}{2F^2} \right) = 1 - \tau\, u^2 + \nu \partial_x (\tau^{-2} \partial_x u), \end{aligned} \tag{1.1}$$

where $\tau = 1/h$ is the reciprocal of fluid height h, u is fluid velocity averaged with respect to height, x is a Lagrangian marker, F is a Froude number given by the ratio between a chosen reference speed of the fluid and speed of gravity waves, and $\nu = R_e^{-1}$, with R_e the Reynolds number of the fluid. The terms 1 and τu^2 on the righthand side of the second equation model, respectively, gravitational force and turbulent friction along the bottom. Roughly speaking, F measures inclination, with $F = 0$ correspond to horizontal and $F \to \infty$ to vertical inclination of the plane.

An interesting and much-studied phenomenon in thin film flow is the appearance of *roll-waves*, or spatially periodic traveling-waves corresponding to solutions

$$(\tau, u)(x, t) = (\bar\tau, \bar u)(x - ct) \tag{1.2}$$

of (1.1). These are well-known *hydrodynamic instabilities,* arising for (1.1) in the region $F > 2$ for which constant solutions, corresponding to parallel flow, are unstable, with applications to landslides, river and spillway flow, and topography of sand dunes and sea beds [BM].

Nonlinear stability of roll-waves themselves has been a long-standing open problem. However, this problem has recently been mostly solved in a series of works by the authors together with Barker, Johnson, and Noble; see [JZN, BJRZ, BJNRZ1, JNRZ, BJNRZ2]. More precisely, it has been shown that, under a certain technical condition having to do with the slope of the traveling-wave profile $(\bar\tau, \bar u)$, *spectral stability* in the sense of Schneider [S1, S2, JZN], *implies linear and nonlinear modulational stability with optimal rates of decay*, and, moreover, asymptotic behavior is well-described by a formal second-order *Whitham equations* obtained by WKB expansion.

Date: March 29, 2015.

Stay of M.R. in Bloomington was supported by French ANR Project no. ANR-09-JCJC-0103-01.

Research of K.Z. was partially supported under NSF grant no. DMS-0300487.

In turn, spectral stability has been characterized analytically in the *weakly unstable* limit $F \to 2$ and numerically for intermediate to large F in terms of two simple power-law descriptions, in the small- and large-F regimes, respectively, of the band of periods X for which roll waves are spectrally stable, as functions of F and discharge rate q (an invariant of the flow describing the flux of fluid through a given reference point) [BJNRZ2]. That is, apart from the technical slope condition, there is at this point a rather complete theory of spectral, linear, and nonlinear stability of roll wave solutions of the St. Venant equations. However, up to now it was not clear whether failure of the slope condition was a purely technical issue or might be an additional mechanism for instability.

Precisely, this slope condition reads, in the Lagrangian formulation (1.1)–(1.2), as

$$2\nu \bar{u}_x < F^{-2}, \tag{1.3}$$

where $\bar{u}$ is the velocity component of traveling wave (1.2). It is seen numerically to be satisfied for $F \lessapprox 3.5$, but to fail for $F \gtrapprox 3.5$ [BJNRZ2]. For comparison, hydraulic engineering applications typically involve Froude numbers $2.5 \lessapprox F \lessapprox 20$ [A, Br1, Br2]; *hence* (1.3) *is a real restriction.*

The role of condition (1.3) in the stability analysis is to obtain pointwise symmetrizability of the linearized equations and thereby high-frequency resolvent bounds and a crucial nonlinear damping estimate used to control higher derivatives in a nonlinear iteration scheme. The purpose of the present brief note is to show, by a refined version of the energy estimates of [JZN, BJRZ], that the pointwise condition (1.3) can be replaced by an *averaged version* that is *always satisfied,* while still retaining the high-frequency resolvent and nonlinear damping estimates needed for the nonlinear analysis of [JZN, JNRZ], thus *effectively completing the nonlinear stability theory.*

The remainder of this paper is devoted to establishing the requisite weighted energy estimates, first, in Sections 2-3, in the simplest, linear time-evolution setting then, in Sections 4.1 and 4.2, respectively, in the closely related high-frequency resolvent and nonlinear time-evolution settings. The estimates so derived may be seen to be periodic-coefficient analogs of weighted "Kawashima-type" estimates derived in the asymptotically-constant coefficient case for the study of stability of large-amplitude viscous shock waves [Z1, Z2, GMWZ], to our knowledge the first examples of such estimates specialized to the periodic setting. We discuss this connection in Sections 5-Section 6. We note, finally, the relation between these weights and the "gauge functions" used for similar purposes in short-time (i.e., well-posedness) dispersive theory [LP, BDD], a connection brought out further by our choice of notation in the proof. This indicates perhaps a potential for wider applications of these ideas in the study of periodic wave trains.

2. Preliminaries

Making the change of variables $x \to x - ct$ to co-moving coordinates, we convert (1.1) to

$$\begin{aligned} &\tau_t - c\tau_x - u_x = 0, \\ &u_t - cu_x + ((2F^2)^{-1}\tau^{-2})_x = 1 - \tau u^2 + \nu(\tau^{-2}u_x)_x, \end{aligned} \tag{2.1}$$

and the traveling-wave solution to a stationary solution $U = \bar{U}(x)$ convenient for stability analyis.

We note for later that the traveling-wave ODE becomes

$$-c\bar{\tau}_x - \bar{u}_x = 0, \qquad \bar{u}_t - c\bar{u}_x + ((2F^2)^{-1}\bar{\tau}^{-2})_x = 1 - \bar{\tau}\bar{u}^2 + \nu(\bar{\tau}^{-2}\bar{u}_x)_x, \tag{2.2}$$

yielding the key fact that

$$f(\bar{\tau})\bar{u}_x = cf(\bar{\tau})\bar{\tau}_x \tag{2.3}$$

is a perfect derivative for any function $f(\cdot)$, hence *zero mean.* We note also as in [JZN] that $c \neq 0$, else $u \equiv$ constant and the equation for τ reduces to first order, hence does not admit nontrivial periodic solutions. Linearizing about $\bar U$ gives the *linearized equations*

$$\begin{aligned} \tau_t - c\tau_x - u_x &= 0,\\ u_t - cu_x - (\alpha)\tau)_x &= \nu(\bar\tau^{-2}u_x)_x - \bar u^2\tau - 2\bar u\bar\tau u, \end{aligned} \tag{2.4}$$

where

$$\alpha := \bar\tau^{-3}(F^{-2} + 2\nu\bar{\bar u}_x). \tag{2.5}$$

With this notation, the slope condition of [JZN] appears as $\bar\tau^3\alpha > 0$. We note that, by (2.3), the mean over one period of $g(\bar\tau)\alpha$ is *positive* for any g:

$$\langle g(\bar\tau)\alpha\rangle = \langle g(\bar\tau)F^{-2}\rangle > 0. \tag{2.6}$$

That is, the slope condition holds always in an averaged sense.[1] We shall show in the rest of the paper that this averaged condition is in fact sufficient for the nonlinear analysis of [JZN, JNRZ].

3. Linear damping estimate

Define now the energy

$$\mathcal{E}(U) := \int \Big(\frac12\phi_1(x)\tau_x^2 + \frac12\phi_2(x)\bar\tau^3u_x^2 + \phi_3(x)\tau u_x\Big). \tag{3.1}$$

A brief computation yields

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(U(t)) = \int \Big(&- (\frac{c}{2}(\phi_1)_x + \alpha\phi_3)\tau_x^2 - (\frac{\nu}{\bar\tau^2}\phi_2)u_{xx}^2 + (\phi_1 - \alpha\phi_2 + \frac{\nu}{\bar\tau^2}\phi_3)\tau_x u_{xx}\Big)\\ &+ O\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big). \end{aligned} \tag{3.2}$$

Taking

$$\frac{c}{2}(\phi_1)_x + \Big(\frac{\alpha\bar\tau^2}{\nu} - \frac{\langle\alpha\bar\tau^2\rangle}{\nu}\Big)\phi_1 = 0, \tag{3.3}$$

$$\phi_1 - \alpha\phi_2 + \frac{\nu}{\bar\tau^2}\phi_3 = 0, \tag{3.4}$$

and

$$0 < \phi_1, \quad 0 < \phi_2 \equiv \text{constant} \ll 1, \tag{3.5}$$

we obtain after another brief computation

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(U(t)) &= \int \Big(-\Big[\frac{\langle\bar\tau^2\alpha\rangle}{\nu}\phi_1 - \frac{\alpha^2\bar\tau^2}{\nu}\phi_2\Big)\tau_x^2 - \Big(\frac{\nu}{\bar\tau^2}\phi_2\Big)u_{xx}^2\Big]\\ &\quad + O\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big)\\ &\le -\eta_1(\|u_{xx}\|_{L^2}^2 + |\tau_x\|_{L^2}^2) + C_1\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big), \end{aligned} \tag{3.6}$$

whence, by Sobolev embedding and the fact that $\mathcal{E}(U) \sim (\|u_x\|_{L^2}^2 + |\tau_x\|_{L^2}^2)$ modulo $\|\tau\|_{L^2}^2$,

$$\frac{d}{dt}\mathcal{E}(U(t)) \le -\eta\mathcal{E}(U(t)) + C\|U(t)\|_{L^2}^2, \tag{3.7}$$

a standard *linear damping estimate.*

Note that, in the last step $\mathcal{E}(U) \sim (\|u_{xx}\|_{L^2}^2 + |\tau_x\|_{L^2}^2)$, we have used in a critical way that $\int\Big(\frac{\alpha\bar\tau^2}{\nu} - \frac{\langle\alpha\bar\tau^2\rangle}{\nu}\Big)$, hence ϕ_1, remains bounded, a consequence of periodicity plus zero mean.

[1] Here and elsewhere we use $\langle h\rangle$ to denote mean over one period of a function h.

4. Applications

4.1. **High-frequency resolvent bound.** Consider now the resolvent equation

$$(\lambda - L)U = F, \qquad U = (\tau, u). \tag{4.1}$$

Letting $\langle \cdot, \cdot \rangle$ denote complex inner product, we find by computations essentially identical to those in Section 3, substituting λU for U_t, that, defining ϕ_j as in (3.3)–(3.5), and

$$\mathcal{E}(U) := \langle \phi_1 u_x, u_x \rangle + \langle \phi_2 \tau_x, \tau_x \rangle + \langle \phi_3 \tau, u_x \rangle,$$

that

$$\begin{aligned} \Re \lambda \mathcal{E}(U) &= \Re(\langle \phi_1 u_x, \lambda u_x \rangle + \langle \phi_2 \tau_x, \lambda \tau_x \rangle + \langle \phi_3 \tau, \lambda u_x \rangle \\ &= \int \Big(-(\frac{c}{2}(\phi_1)_x + \alpha \phi_3) \tau_x^2 - (\frac{\nu}{\bar\tau^2}\phi_2) u_{xx}^2 + (\phi_1 - \alpha\phi_2 + \frac{\nu}{\bar\tau^2}\phi_3)\tau_x u_{xx} \Big) \\ &\quad + O\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big) + O(\|U_x\|_{L^2}\|F_x\|_{L^2}) \\ &\le -\eta \mathcal{E}(U) + C(\|U\|_{L^2}^2 + \|F\|_{H^1}^2). \end{aligned} \tag{4.2}$$

Combining (4.2) with the with easy estimate $|\lambda| \|U\|_{L^2}^2 = |\langle U, \lambda U \rangle| \le C(\|U\|_{H^1}^2 + \|F\|_{L^2}^2)$, or

$$\|U\|_{L^2}^2 \le C|\lambda|^{-1}(\|U\|_{H^1}^2 + \|F\|_{L^2}^2),$$

we get for $|\lambda| \gg 1$, the estimate $(\Re\lambda + \eta)\|U\|_{H^1}^2 \le \|F\|_{H^1}^2$, yielding a uniform resolvent bound

$$\|U\|_{H^1}^2 \le C \|F\|_{H^1}^2$$

for $\Re\lambda \ge -\eta/2$ and $|\lambda|$ sufficiently large, as required in the analysis of [JZN, JNRZ].

4.2. **Nonlinear damping estimate.** Following [JZN, JNRZ], introduce the perturbation variable

$$V(x,t) = \tilde U(x - \psi(x,t), t) - \bar U(x) \tag{4.3}$$

where $\tilde U(x,t) = (\tilde\tau, \tilde u)(x,t)$ is a second solution of the St. Venant equations close to the background profile $\bar U = (\bar\tau, \bar u)$, and the real-valued function $\psi(x,t)$ is a phase shift to be determined together with V. A key identity established in [JZN, JNRZ] is

$$(\partial_t - L)\,(V + \psi \bar U_x) = \partial_x \mathcal{N} + \partial_t (V\psi_x) - \partial_x (V \psi_t),$$

where $\mathcal{N} = O((|V, V_x, \psi_{x,t}||V, \psi_{x,t}|)$ so long as V and $\psi_{x,t}$ remain small. Recalling that $L\bar U_x = 0$, $\partial_t \bar U_x = 0$, we find that we may rewrite this as

$$((1+\psi_x)\partial_t - L)\, V = \partial_x \mathcal{N}_2 + \mathcal{M}_2, \tag{4.4}$$

where $\mathcal{N}_2 = O((|V, u_x, \psi_{x,t}||V, \psi_{x,t}|)$, $\partial_x \mathcal{N}_2 = O((|V_x, u_{xx}, \partial_x\psi_{x,t}||V, \psi_{x,t}|)$ and $\mathcal{M}_2 = O(|\psi_{x,t}, \psi_{xx}|)$, $\partial_x \mathcal{M}_2 = O(|\psi_{x,t}, \psi_{xx}|) + O(|\partial_x \psi_{x,t}, \psi_{xxx}|)$, so long as V and $\psi_{x,t}$ remain small.

Defining the modified energy

$$\mathcal{E}_2(U) := \int (1+\psi_x)\Big(\frac{1}{2}\phi_1(x)\tau_x^2 + \frac{1}{2}\phi_2(x)\bar\tau^3 u_x^2 + \phi_3(x)\tau u_x \Big), \tag{4.5}$$

repeating the argument of Section 3, absorbing nonlinear terms into the linear ones and separating out $\psi_{x,t}$ terms using Young's inequality, we obtain in analogy to (3.7), the nonlinear estimate

$$\frac{d}{dt}\mathcal{E}_2(V) \le -\eta \mathcal{E}_2(V) + C\left(\|V(t)\|_{L^2}^2 + \|(\psi_t, \psi_x)(t)\|_{H^1}^2 \right), \quad \eta, C > 0.$$

Differentiating the equations and performing the same estimate on $\partial_x^k V$, we obtain likewise

$$\frac{d}{dt}\mathcal{E}_2(\partial_x^k V) \le -\eta \mathcal{E}_2(\partial_x^k V) + C\left(\|V(t)\|_{L^2}^2 + \|(\psi_t, \psi_x)(t)\|_{H^k}^2 \right), \quad \eta, C > 0, \tag{4.6}$$

so long as $\|v,\psi_{x,t}\|_{H^k}$ remains sufficiently small. Applying Gronwall's inequality and recalling that $\mathcal{E}(\partial_x^k V)\sim\|\partial_x^k V)\|_{L^2}^2$ modulo lower-derivative terms, we obtain the following key estimate showing that higher Sobolev norms $\|V\|_{H^k}$ are slaved to $\|V\|_{L^2}$ and $\|\psi_{x,t}\|_{H^k}$, the final estimate needed for the analysis of [JZN, JNRZ], *without assuming the slope condition* (1.3).

Proposition 4.1 (Nonlinear damping)**.** *There exist $\eta, C>0$ such that if V and ψ solve* (4.4) *on $[0,T]$ for $T>0$ such that $\sup_{t\in[0,T]}\|(V,\psi_x)(t)\|_{H^K(\mathbb{R})}+\sup_{t\in[0,T]}\|\psi_t(t)\|_{H^{K-1}(\mathbb{R})}$ is sufficiently small, then, for all $0\le t\le T$,*

$$\begin{aligned}\|V(t)\|_{H^K(\mathbb{R})}^2 &\le Ce^{-\theta t}\|V(0)\|_{H^K(\mathbb{R})}^2\\ &\quad + C\int_0^t e^{-\theta(t-s)}\left(\|V(s)\|_{L^2(\mathbb{R})}^2+\|(\psi_t,\psi_x,\psi_{xx})(s)\|_{H^{K-1}(\mathbb{R})}^2\right)ds.\end{aligned}$$

4.3. **Stability and asymptotic behavior.** We state for completeness the resulting theorem on nonlinear stability and asymptotic behavior. For Bloch operators $L_\xi := e^{-i\xi x}Le^{i\xi x}$ acting on periodic period-one functions, $\xi\in[-\pi,\pi)\subset\mathbb{R}$, *diffusive spectral stability* [JZN, JNRZ, S1] is:

(D1) L_0 has zero-eigenspace of dimension 2 at $\lambda=0$ and no other pure imaginary eigenvalues.
(D2) $\Re\sigma(L_\xi)\le-\theta|\xi|^2$ for some constant $\theta>0$.

Parametrizing the traveling-wave solutions (up to translation) near $\bar U$ by

$$\bar U^{k,M}(kx+\omega(M,k)t),$$

where M is the mean of $\bar u^{M,k}$ over one period and k is the wave number, we obtain by WKB expansion $U(x,t)\sim\bar U^{(M,k)(x,t)}(\Psi(x,t))$, where (M,k) satisfy the associated formal *second-order Whitham equations*

$$\begin{aligned} M_t+F_x &= (d_1(M,k)M_x+e_1(M,K)k_x)_x,\\ k_t+\omega_x &= (d_2(M,k)M_x+e_2(M,K)k_x)_x,\end{aligned}\tag{4.7}$$

where $k\sim\Psi_x$, F is the mean over one period of $F(\bar U^{M,k}$, $\omega=\omega(M,k)$ as above is the nonlinear dispersion relation associated with the profile existence problem, and d_j, e_j are coefficients obtained by more complicated higher-order compatibility conditions. For derivations, see [W, Se, NR, JNRZ].

Then, from the estimates just derived, combined with the previous analysis of [JNRZ], we obtain the following comprehensive result on nonlinear stability and asymptotic behavior of roll waves.

Theorem 4.2 (Asymptotic behavior)**.** *Let $\eta>0$, arbitrary, and $K\ge 4$. Under the assumptions (D1)–(D2) and $E_0:=\|\tilde U-\bar U(\cdot+h\|_{L^1\cap H^5}|_{t=0}+\|h_x\|_{L^1\cap H^5}\ll 1$, and suitable parametrization, there exist $M(x,t)$, and $\psi(x,t)$ such that $\psi(\cdot,0)=h_0$ and, with global phase shift $\psi_\infty=(h_0(-\infty)+h_0(\infty))/2$, for $t\ge 0$, $2\le p\le\infty$,*

$$\begin{aligned}\|\tilde U(\cdot-\psi(\cdot,t),t)-U^{\bar M+M(\cdot,t),k_*/(1-\psi_x(\cdot,t))}(\cdot)\|_{L^p(\mathbb{R})} &\lesssim E_0\ln(2+t)\,(1+t)^{-\frac34},\\ \|(k_*\psi_x,M)(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac12(1-1/p)},\\ \|\psi(t)-\psi_\infty\|_{L^\infty(\mathbb{R})} &\lesssim E_0.\end{aligned}\tag{4.8}$$

Moreover, setting $\Psi(\cdot,t)=(\mathrm{Id}-\psi(\cdot,t))^{-1}$, $\kappa=k_*\partial_x\Psi$, $\mathcal{M}(\cdot,t)=(\bar M+M(\cdot,t))\circ\Psi(\cdot,t)$, and defining $(\mathcal{M}_W,\kappa_W)$ to be solutions of (4.7) *with initial data*

$$\begin{aligned}\mathcal{M}_W(\cdot,0) &= \bar M+\tilde U_0-\bar U\circ\Psi(\cdot,0)+\left(\frac{1}{\partial_x\Psi(\cdot,0)}-1\right)\left(\bar U\circ\Psi(\cdot,0)-\bar M\right),\\ \kappa_W(\cdot,0) &= k_*\partial_x\Psi(\cdot,0),\quad \Psi_W(\cdot,0)\;=\;\Psi(\cdot,0),\end{aligned}\tag{4.9}$$

we have, for $t \geq 0$, $2 \leq p \leq \infty$,

$$
\begin{aligned}
\|(\mathcal{M},\kappa)(t) - (\mathcal{M}_W,\kappa_W)(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta},\\
\|\Psi(t) - \Psi_W(t)\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)+\eta};
\end{aligned} \tag{4.10}
$$

in particular, $\kappa = k_*\partial_x\Psi$, $\kappa_W = k_*\partial_x\Psi_W$, and

$$
\begin{aligned}
\|\tilde U(\cdot,t) - U^{\mathcal{M}(\cdot,t),\kappa(\cdot,t)}(\Psi(\cdot,t))\|_{L^p(\mathbb{R})} &\lesssim E_0 \ln(2+t)\,(1+t)^{-\frac{3}{4}},\\
\|\tilde U(\cdot,t) - U^{\mathcal{M}_W(\cdot,t),\kappa_W(\cdot,t)}(\Psi_W(\cdot,t))\|_{L^p(\mathbb{R})} &\lesssim E_0(1+t)^{-\frac{1}{2}(1-1/p)+\eta}.
\end{aligned} \tag{4.11}
$$

5. The shock wave case

We next turn to the connection with viscous shock theory, showing that the same linear damping estimate (3.7) may be obtained by essentially the same argument in the asymptotically-constant, viscous shock wave case, thus recovering the bounds established in [Z1, Z2, GMWZ] by related but slightly different weighted Kawashima-type energy estimates.[2] The equations of isentropic gas dynamics in Lagrangian coordinates, expressed in a comoving frame are

$$
\begin{aligned}
&\partial_t\tau - c\partial_x\tau - \partial_x u = 0,\\
&\partial_t u - c\partial_x u + \partial_x p(\tau) = \nu\partial_x(\tau^{-1}\partial_x u),
\end{aligned} \tag{5.1}
$$

where τ is specific volume, u is velocity, and p is pressure.

Traveling waves $(\tau,u)(x,t) = (\bar\tau,\bar u)(x)$ satisfy the profile ODE

$$-c^2\tau - p(\tau) + q = cu'/\tau, \qquad q = \text{constant}.$$

We note as in the periodic case that $c \neq 0$, else $u, p(\tau) \equiv$ constant, yielding $\tau \equiv$ constant, a trivial solution. Assume that the shock is noncharacteristic, i.e., $-p'(\tau_\pm) \neq c^2$, hence $\tau_\pm$ are nondegenerate equilibria and the shock profile decays exponentially to its endstates as $x \to \pm\infty$.

The linearized equations are

$$\tau_t - c\tau_x - u_x = 0, \qquad u_t - cu_x - (\alpha)\tau)_x = \nu(\bar\tau^{-1}u_x)_x, \tag{5.2}$$

where $\alpha := p'(\bar\tau) + \nu\frac{\bar u_x}{\bar\tau}$. Define $I\big(\frac{\alpha\bar\tau^2}{\nu}\big)$ to be a smooth interpolant between $\frac{\alpha\bar\tau^2}{\nu}|_{x=\pm\infty}$ such that

$$\frac{\alpha\bar\tau^2}{\nu} - I\big(\frac{\alpha\bar\tau^2}{\nu}\big) = O(e^{-\theta|x|}). \tag{5.3}$$

Taking as before $\mathcal{E}(U) := \int\Big(\frac{1}{2}\phi_1(x)\tau_x^2 + \frac{1}{2}\phi_2(x)\bar\tau^3u_x^2 + \phi_3(x)\tau u_x\Big)$, we find again

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(U(t)) = \int\Big(&-(\frac{c}{2}(\phi_1)_x + \alpha\phi_3)\tau_x^2 - (\frac{\nu}{\bar\tau^2}\phi_2)u_{xx}^2 + (\phi_1 - \alpha\phi_2 + \frac{\nu}{\bar\tau^2}\phi_3)\tau_x u_{xx}\Big)\\
&+ O\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big).
\end{aligned} \tag{5.4}
$$

Taking $\frac{c}{2}(\phi_1)_x + \Big(\frac{\alpha\bar\tau^2}{\nu} - I\big(\frac{\alpha\bar\tau^2}{\nu}\big)\Big)\phi_1 = 0$, $\phi_1 - \alpha\phi_2 + \frac{\nu}{\bar\tau^2}\phi_3 = 0$, $0 < \phi_1$, and $0 < \phi_2 \equiv$ constant $\ll 1$, we thus have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(U(t)) = \int\Big(&-\Big[I(\frac{\bar\tau^2\alpha}{\nu})\phi_1 - \frac{\alpha^2\bar\tau^2}{\nu}\phi_2\Big)\tau_x^2 - \Big(\frac{\nu}{\bar\tau^2}\phi_2\Big)u_{xx}^2\Big]\\
&+ O\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big)\\
\leq\ & -\eta_1(\|u_{xx}\|_{L^2}^2 + |\tau_x\|_{L^2}^2) + C_1\big(\|u\|_{H^2} + \|\tau\|_{H^1})(\|u\|_{H^1} + \|\tau\|_{L^2})\big),
\end{aligned} \tag{5.5}
$$

---

[2]The weights used in [Z1, Z2, GMWZ] are effectively $\phi_1 = \phi_2 >> \phi_3$, $(\phi_1)_x = -C\Big(\frac{\alpha\bar\tau^2}{\nu} - I\big(\frac{\alpha\bar\tau^2}{\nu}\big)\Big)\phi_1$, $C >> 1$.

and thereby the same linear damping estimate as in the periodic-coefficient case:

$$\frac{d}{dt}\mathcal{E}(U(t)) \le -\eta\mathcal{E}(U(t)) + C\|U(t)\|_{L^2}^2. \tag{5.6}$$

As in the periodic case, a crucial point is that $\int \Big(\frac{\alpha\bar\tau^2}{\nu} - I\big(\frac{\alpha\bar\tau^2}{\nu}\big)\Big)$, hence ϕ_1, remains bounded, so that $\mathcal{E}(U) \sim (\|u_{xx}\|_{L^2}^2 + |\tau_x\|_{L^2}^2)$ modulo $\|\tau\|_{L^2}^2$, a property following in this case by exponential decay, (5.3).

6. Discussion: relation to the asymptotically-constant case

The above may be recognized as exponentially weighted Kawashima-type estimates similar to that used in the study of viscous shock stability in [Z1, Z2, GMWZ], reflecting the growing analogy the periodic and asymptotically constant case, as evidenced for example by the relation between Floquet's Lemma on periodic coordinate transformation of periodic- to constant-coefficient systems of equations vs. the Conjugation Lemma of [MZ] on asymptotically constant-coefficient coordinate transformations from asymptotically constant- to constant-coefficient systems.

For a general hyperbolic-parabolic principal part $U_t + AU_x = (BU_x)_x$, $\Re B \ge 0$, a Kawashima-type estimate is on an energy $\mathcal{E}(U) := \langle U, A^0 U\rangle + \langle UK\partial_x, U\rangle + \langle U_x, A^0 U_x\rangle$, where A^0 is symmetric positive definite and K is skew symmetric, chosen, where possible, so that

$$\Re(A^0 B + KA) > 0. \tag{6.1}$$

For small-amplitude shocks, this may typically be achieved globally, but for large-amplitude waves it can be done typically only near $x \to \pm\infty$ where A is symmetrizable.

A key to the treatment of large-amplitude viscous shock waves, first carried out in [MaZ], motivated by some clever "transverse" energy estimates of Goodman [G] in the study of small-amplitude stability, is to introduce an asymptotically constant exponential weight in the energy:

$$\mathcal{E}_2(U) := \langle U, \phi A^0 U\rangle + \langle U\phi K\partial_x, U\rangle + \langle U_x, \phi A^0 U_x\rangle,$$

recovering coercivity in the near field $|x| \le C$ where (6.1) fails. In the present case,

$$A = \begin{pmatrix} -c & -1 \\ \alpha & -c \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^0 = \mathrm{diag}$$

for either of the St. Venant or isentropic compressible Navier–Stokes equations. The issue in the latter case is that symmetrizability holds in general only in the limits $x \to \pm\infty$, in the former that it holds only on average, but not pointwise. In either case, we have seen that energy estimates can be recovered by the use of appropriate asymptotically-constant, or periodic exponential weights.

References

[A] M.B. Abd-el-Malek, *Approximate solution of gravity-affected flow from planar sluice gate at high Froude number,* J. Computational and Appl. Math. 35 (1991) 83-97.

[BM] N.J. Balmforth and S. Mandre, *Dynamics of roll waves,* J. Fluid Mech. 514 (2004), 1–33.

[BJNRZ1] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *Whitham averaged equations and modulational stability of periodic solutions of hyperbolic-parabolic balance laws,* Journées équations aux dérivées partielles (2011), Exp. No. 3.

[BJNRZ2] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, *$F \to 2$,* TODO.

[BDD] *Well-posedness of one-dimensional Korteweg models,* Electronic Journal of Differential Equations, 59, 1-35 (2006).

[Br1] R.R. Brock, *Development of roll-wave trains in open channels.* J. Hydraul. Div., Am. Soc. Civ. Eng. 95:4 (1969) 1401–1428.

[Br2] R.R. Brock, *Periodic permanent roll waves.* J. Hydraul. Div., Am. Soc. Civ. Eng. 96:12 (1970) 2565–2580.

[GMWZ] O. Guès, G. Métivier, M. Williams, and K. Zumbrun, *Navier-Stokes regularization of multidimensional Euler shocks,* Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 75–175.

[JNRZ] M. A. Johnson, P. Noble, L.M. Rodrigues, K. Zumbrun *Behaviour of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations.*, Inventiones Math., 197 (2014) no. 1, 115-213.

[BJRZ] B. Barker, M. Johnson, M. Rodrigues, and K. Zumbrun, *Metastability of Solitary Roll Wave Solutions of the St. Venant Equations with Viscosity,* Phys. D 240(16):1289-1310, 2011.

[G] J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

[JZN] M. Johnson, K. Zumbrun, and P. Noble, *Nonlinear stability of viscous roll waves,* Nonetheless, as pointed out first in SIAM J. Math. Anal. 43 (2011), no. 2, 557-611.

[NR] P. Noble, L.-M. Rodrigues *Whitham's Modulation Equations and Stability of Periodic Wave Solutions of the Generalized Kuramoto-Sivashinsky Equations*, Indiana Univ. Math. J. 62 (2013), no. 3, 753–783.

[LP] W. K. Lim and G. Ponce, *On the initial value problem for the one dimensional quasi-linear Schrödinger equations.* SIAM J. Math. Anal., 34(2):435–459 (electronic), 2002.

[MaZ] C. Mascia and K. Zumbrun, *Stability of large-amplitude shock profiles of hyperbolic–parabolic systems.* Arch. Rat. Mech. Anal. 172 (2004), no. 1, 93–131.

[MZ] G. Métivier and K. Zumbrun, *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems,* Mem. Amer. Math. Soc. 175 (2005), no. 826, vi+107 pp.

[S1] G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions– abstract theorem and higher space dimensions*, Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.

[S2] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation,* (English. English summary) Comm. Math. Phys. 178 (1996), no. 3, 679–702.

[Se] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis*, Comm. Partial Differential Equations 30 (2005), no. 1-3, 259–282.

[W] G. B. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[Z1] K. Zumbrun, *Stability of large-amplitude shock waves of compressible Navier-Stokes equations.* With an appendix by Helge Kristian Jenssen and Gregory Lyng. Handbook of mathematical fluid dynamics. Vol. III, 311–533, North-Holland, Amsterdam, (2004).

[Z2] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity.* Hyperbolic systems of balance laws, 229–326, Lecture Notes in Math., 1911, Springer, Berlin, 2007.

Université Lyon 1, Villeurbanne, France
E-mail address: rodrigues@math.univ-lyon1.fr

Indiana University, Bloomington, IN 47405
E-mail address: kzumbrun@indiana.edu