Stability of noncharacteristic boundary layers in the standing-shock limit

Kevin Zumbrun*

September 15, 2008

Abstract

We investigate one- and multi-dimensional stability of noncharacteristic boundary layers in the limit approaching a standing planar shock wave $\bar U(x_1)$, $x_1 > 0$, obtaining necessary conditions of (i) weak stability of the limiting shock, (ii) weak stability of the constant layer $u \equiv U_- := \lim_{z\to -\infty} \bar U(z)$, and (iii) nonnegativity of a modified Lopatinski determinant similar to that of the inviscid shock case. For Lax 1-shocks, we obtain equally simple sufficient conditions; for p-shocks, $p > 1$, the situation appears to be more complicated. Using these results, we determine stability of certain gas dynamical boundary-layers, generalizing earlier work of Serre–Zumbrun and Costanzino–Humphreys–Nguyen–Zumbrun.

Contents

*Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grants number DMS-0070765 and DMS-0300487.

1 Introduction

Consider a boundary layer, or stationary solution,

$$\tilde U = \bar U(x_1), \quad \lim_{z\to+\infty} \bar U(z) = U_+, \quad \bar U(0) = U_0 \tag{1.1}$$

of a hyperbolic–parabolic system of conservation laws on the quarter-space

$$\tilde U_t + \sum_j F_j(\tilde U)_{x_j} = \sum_{jk} (B_{jk}(\tilde U)\tilde U_{x_k})_{x_j}, \quad x \in \mathbb{R}^d_+ = \{x_1 > 0\}, \quad t > 0, \tag{1.2}$$

$\tilde U, F_j \in \mathbb{R}^n$, $B_{jk} \in \mathbb{R}^{n\times n}$,

$$\tilde U = \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b^1_{jk} & b^2_{jk} \end{pmatrix}, \quad \tilde u \in \mathbb{R}^{n-r},\ \tilde v \in \mathbb{R}^r,$$

with initial data $\tilde U(x,0) = \tilde U_0(x)$ and boundary conditions as specified in (1.6) below, that is noncharacteristic both in the hyperbolic sense

$$\det dF^1(\bar U_+) \neq 0$$

and with respect to the original (partially) parabolic problem as described in (H1)–(H3) below. Such layers occur, for instance, in gas- and magnetohydrodynamics with inflow or outflow boundary conditions, for example in flow around an airfoil with micro-suction or blowing; see [GMWZ5, YZ, NZ1, NZ2] for further discussion.

As for any gas-dynamical flow, an important question is *stability* of these solutions under perturbation of the initial or boundary data. This question has been investigated in [GR, MZ1, GMWZ5, GMWZ6, YZ, NZ1, NZ2] for arbitrary-amplitude boundary-layers using Evans function techniques, with the result that under quite general circumstances (see model assumptions below) linearized and nonlinear stability reduce to a generalized spectral stability condition phrased in terms of the *Evans function*, a Wronskian associated with the family of eigenvalue ODE obtained by Fourier transform in the transverse directions $\tilde{x} := (x_2, \dots, x_d)$. See also the small-amplitude results of [GG, R3, MN, KNZ, KK] obtained by energy methods.

The Evans function is readily evaluable numerically; see, e.g., [CHNZ, HLyZ1, HLyZ2]. As pointed out in [SZ, CHNZ, GMWZ5], it is also evaluable analytically in certain interesting asymptotic limits. For example, it was shown in [GMWZ5] that the Evans function converges in the small-amplitude limit as $\bar{U}$ approaches the constant layer $U \equiv U_+$ to the Evans function of the constant layer, uniformly on compact sets of frequencies $\Re\lambda \geq 0$, and, as a consequence, *stability of small-amplitude layers is determined by stability of the limiting constant layer*. This result was used in turn to show that noncharacteristic boundary layers of general symmetric–dissipative systems (defined below) are spectrally stable in the small-amplitude limit.

A different asymptotic limit considered for special cases in [SZ, CHNZ] is the *standing shock* limit $X \to +\infty$ in the case

$$\bar{U}^X(x) = \hat{U}(x_1 - X), \quad \lim_{z \to \pm\infty} \hat{U}(z) = U_\pm \tag{1.3}$$

that $\bar{U}$ is the restriction to $x_1 > 0$ of a standing shock solution $\hat{U}(\cdot - X)$. It is natural to guess that there might be some relation between boundary-layer stability in this limit and stability of the limiting shock wave, and indeed it was shown in [CHNZ] for the case of isentropic ideal gas dynamics that the boundary layer Evans function, suitably normalized, converges in the standing-shock limit to the Evans function of the limiting shock wave on compact subsets of frequencies $\Re\lambda \geq 0$, in complete analogy with the small-amplitude limit. On the other hand, it was shown in [SZ] for the case of full (nonisentropic) ideal gas dynamics that boundary layers can in some parameter regimes be unstable in the standing-shock limit despite stability (see [HLyZ1]) of the limiting shock wave.

In the present paper, we revisit the standing-shock limit in the general case, obtaining a result subsuming and illuminating these previous ones. Moreover, we carry out our investigations in multi-dimensions, whereas the analyses of [SZ, CHNZ] were specific to the one-dimensional case. Specifically, we show that the boundary-layer Evans function, suitably normalized, converges in the standing-shock limit, uniformly on compact subsets of frequencies $\eta \in \mathbb{R}^{d-1}$, $\Re\lambda \geq 0$, to the product of the Evans function of the limiting shock wave and the Evans function of the constant layer $U \equiv U_-$ at the left endstate of the shock. For symmetric–dissipative systems, this implies by stability of constant layers that the Evans function can be further renormalized so as to converge simply to the Evans function of the limiting shock, similarly as was shown in [CHNZ] in the special one-dimensional isentropic ideal gas case.

A consequence is that stability of both the limiting shock and the constant layer $U \equiv U_-$ are *necessary conditions* for stability of boundary layers in the standing-shock limit. On the other hand, these are not sufficient, as even stable shock waves have an (one-dimensional) eigenvalue at $\eta = 0$, $\lambda = 0$ due to translation-invariance, whereas stable boundary layers do not. A further necessary condition, therefore, is nonnegativity of the *stability index* (defined below) counting parity of the number of (one-dimensional) unstable roots $\eta = 0$, $\Re\lambda > 0$, indicating that the zero eigenvalue of the limiting shock does not perturb into the positive half-plane. Negativity of the stability index in the standing-shock limit, indicating an odd number of unstable eigenvalues, was what was shown in [SZ] in order to obtain one-dimensional instability.

Here, we develop these ideas substantially further, determining for Lax 1-shocks a simple and general stability determinant $\hat\Delta(\tilde\xi, \lambda, \eta)$ extending the Lopatinski determinant $\Delta(\tilde\xi, \lambda)$ of the inviscid shock case, $\tilde\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$, $\lambda \in \mathbb{C}$, with $\Re\lambda$, $\eta \geq 0$, for which nonvanishing on the strictly positive half-space $\Re\lambda$, $\eta > 0$ is necessary and nonvanishing on the nonnegative half-space $\Re\lambda$, $\eta \geq 0$ together with stability of the limiting shock $\hat U$ and the constant layer $U \equiv U_-$ is sufficient for stability in the standing-shock limit, in one- and multi-dimensions. We then use this condition to investigate stability in various interesting situations. For p-shocks, $p > 1$, the situation is considerably more tricky, involving a complicated double limit.

1.1 Equations and assumptions

Consider a family $\bar U^X(x)$ of boundary-layers (1.3) of (1.2) consisting of translations of a standing shock solution $\hat U$. Following [GMWZ5, GMWZ6], we assume that the conservation law (1.2) can be rewritten in nonconservative form, after an invertible change of variables $\tilde U \to \tilde W$, as a quasilinear hyperbolic–parabolic system

$$\tilde A_0(\tilde W)u_t + \sum_{j=1}^{d} \tilde A_j(\tilde W)\partial_j(\tilde W) - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(\tilde B_{jk}(\tilde W)\partial_k \tilde W\big) = 0, \tag{1.4}$$

$\tilde A_0$ invertible, with block structure

$$\tilde A_0(\tilde W) = \begin{pmatrix} \tilde A_0^{11} & 0 \\ \tilde A_0^{21} & \tilde A_0^{22} \end{pmatrix}, \quad \tilde B_{jk}(\tilde W) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde B_{jk}^{22} \end{pmatrix}, \tag{1.5}$$

a corresponding splitting $\tilde W = (\tilde w^1, \tilde w^2) \in \mathbb{R}^{n-r} \times \mathbb{R}^r$, and decoupled boundary conditions

$$\left\{ \begin{array}{l} \Upsilon_1(\tilde w^1)_{|x\in\partial\Omega} = g_1(t,x), \\ \Upsilon_2(\tilde w^2)_{|x\in\partial\Omega} = g_2(t,x), \\ \Upsilon_3(\tilde w, \partial_{x_1}\tilde w^2, \partial_{\tilde x}\tilde w^2)_{|x\in\partial\Omega} = 0, \end{array} \right. . \tag{1.6}$$

where $\Upsilon_3(\tilde w, \partial_{x_1}\tilde w^2, \partial_{\tilde x}\tilde w^2) = K_1\partial_{x_1}\tilde w^2 + \sum_{j=2}^d K_j(\tilde w)D_{x_j}\tilde w^2$, $K_1 \equiv$ constant, $\dim\Upsilon_1 = n-r$ in the inflow case and 0 in the outflow case (defined in (H2) just below), and

$$\dim\Upsilon_2 + \dim\Upsilon_3 = r.$$

We make the following technical hypotheses following [Z1, Z3, GMWZ5].

(H0) $F^j, B^{jk}, \tilde A^0, \tilde A^j, \tilde B^{jk}, \tilde W(\cdot), \tilde g(\cdot,\cdot) \in C^s$, $s \geq 2$.

(H1) The eigenvalues of $\sum_j (\tilde A_0^{11})^{-1} \tilde A_j^{11} \xi_j$ are real and semisimple for all $\xi \neq 0$ in $\mathbb{R}^d$.

(H2) The eigenvalues of $(\tilde A_0^{11})^{-1} \tilde A_1^{11}$ are either strictly positive or strictly negative, that is, either $\sigma(\tilde A_0^{11})^{-1} \tilde A_1^{11} \geq \theta_1 > 0$ (inflow case) or $\sigma(\tilde A_0^{11})^{-1} \tilde A_1^{11} \leq -\theta_1 < 0$ (outflow case).

(H3) $\Re \sigma \sum_{jk} b_2^{jk} \xi_j \xi_k \geq \theta |\xi|^2 > 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$.

(H4) The eigenvalues of $\sum_j dF_\pm^j \xi_j$ are real, semisimple, and have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

(H5) The eigenvalues of $dF^1(U_\pm)$ are nonzero.

(H6) $\Re \sigma \left(\sum_j i \xi_j dF^j(U_\pm) - \sum_{j,k} \xi_j \xi_k B^{jk}(U_\pm) \right) \leq -\theta \frac{|\xi|^2}{1+|\xi|^2}$ for all $\xi \in \mathbb{R}^d$, some $\theta > 0$.

Definition 1.1. *The system* (1.2), (1.4) *is symmetric dissipative at* $U_\pm$ *if in a neighborhood of* $U_\pm$ *there exists a real matrix* $S(\tilde U)$ *depending smoothly on* $\tilde U$ *such that for such that for all* $\xi \in \mathbb{R}^d \backslash \{0\}$ $S(\tilde U) \tilde A_0(\tilde U)$ *is symmetric definite positive and block-diagonal,* $S(\tilde U) \sum_j \tilde A_j(\tilde U) \xi_j$ *is symmetric, and* $\Re S(\tilde U) \sum \tilde B_{jk}(\tilde U) \xi_j \xi_k$ *is nonnegative with kernel of dimension* $n - r$.

Alternative Hypothesis H4′. For systems that are symmetric dissipative at $U_\pm$, we may relax (H4) to:

(H4') About each $\xi \in \mathbb{R}^d \backslash \{0\}$, the eigenvalues of $\sum_j dF_\pm^j \xi_j$ (necessarily real and semisimple, by symmetrizability) are either of constant multiplicity or else are totally nonglancing in the sense of [GMWZ6], Definition 4.3.

Definition 1.2. *The system* (1.2), (1.4) *is symmetric hyperbolic–parabolic if there exists a real matrix* $S(\tilde U)$ *depending smoothly on* $\tilde U$ *such that for all* $\xi \in \mathbb{R}^d \backslash \{0\}$ *the matrix* $S(\tilde U) \tilde A_0(\tilde U)$ *is symmetric positive definite and block-diagonal,* $(S(\tilde U) \sum_j \tilde A_j(\tilde U) \xi_j)^{11}$ *is symmetric, and the symmetric matrix* $\Re S(\tilde U) \sum \tilde B_{jk}(\tilde U) \xi_j \xi_k$ *is nonnegative with kernel of dimension* $n - r$.

Examples 1.3. Hypotheses (H1)–(H6) are satisfied for standing shocks of the compressible Navier–Stokes equations with van der Waal equation of state, yielding boundary layers for which the normal velocity of the fluid is nonvanishing at U_0. This corresponds to the situation of a porous boundary through which fluid is pumped in or out, in contrast to the characteristic, no-flux boundary conditions encountered at a solid material interface for which normal velocity is set to zero. See [YZ, GMWZ5, NZ1, NZ2] for further discussion of this situation and applications to aerodynamics.

Hypotheses (H1)-(H6) with (H4) replaced by (H4') are satisfied for extreme (i.e., 1- or n-family) standing Lax shocks of the viscous MHD equations with van der Waal equation of state, with similar velocity restrictions on the plasma at U_0, but fail for intermediate shocks. Hypotheses (H1)–(H6) are generically satisfied for viscous MHD in dimension one, but fail always for viscous MHD in dimensions greater than or equal to two; see [MZ2, GMWZ5, GMWZ6] for further discussion. Both gas dynamics and MHD equations with van der Waals equation of state are symmetric hyperbolic–parabolic systems that are symmetric dissipative at $U_\pm$ for standing shocks connecting thermodynamically stable endstates [Z3, GMWZ4].

Finally, regarding the standing shock $\hat{U}$, ordering the eigenvalues of $dF(U_\pm)$ as

$$a_1^\pm < a_2^\pm \cdots < a_n^\pm,$$

we assume:

(H7) Profile $\hat{U}$ is a transversal viscous Lax p-shock, i.e.,

$$a_{p-1}^- < 0 < a_p^-, \quad a_p^+ < 0 < a_{p+1}^+ \tag{1.7}$$

and $\hat{U}$ is a transversal connection of the standing wave ODE with boundary conditions $\tilde{U}_\pm$ (see [MaZ3] for a detailed discussion of the standing wave ODE).

The eigenvalues $a_j^\pm$ correspond to characteristic speeds at $U_\pm$ of the associated one-dimensional inviscid system $U_t + F_1(U)_{x_1} = 0$, with a_p the principal characteristic speed associated with the shock.

1.2 The Evans condition

The linearized eigenvalue equations of (1.2), (1.6) about $\bar{U}$ are

$$\lambda U = LU := \sum_{j,k} (B_{jk} U_{x_k})_{x_j} - \sum_j (A_j U)_{x_j}, \tag{1.8}$$

with homogeneous boundary conditions

$$\Upsilon'(W, \partial_{\tilde{x}} w^2, \partial_{x_1} w^2)_{|x_1=0} = 0 \tag{1.9}$$

expressed in linearized $\tilde{W}$-coordinates $W := (\partial \tilde{W}/\partial \tilde{U})(\bar{U})U$, where W and U denote perturbations of $\tilde{W}$ and $\tilde{U}$.

Taking the Fourier transform in $\tilde{x} := (x_2, \ldots, x_d)$, we obtain a family of eigenvalue ODE

$$\begin{aligned} \lambda U = L_{\tilde{\xi}} U := \overbrace{(B_{11}U')' - (A_1 U)'}^{L_0 U} - i \sum_{j \neq 1} A_j \xi_j U + i \sum_{j \neq 1} B_{j1} \xi_j U' \\ + i \sum_{k \neq 1} (B_{1k} \xi_k U)' - \sum_{j,k \neq 1} B_{jk} \xi_j \xi_k U \end{aligned} \tag{1.10}$$

with boundary conditions

$$\Upsilon'(W, i\tilde{\xi} w^2, \partial_{x_1} w^2)_{|x_1=0} = 0. \tag{1.11}$$

1.2.1 The boundary-layer Evans function

A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re\lambda > 0$ of the linearized operator L about the wave. As described in Section 2, this is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re\lambda > 0$ of the *Evans function*

$$D(\tilde{\xi}, \lambda)$$

a Wronskian associated with (1.10) with columns consisting of bases of the subspace of solutions decaying as $x_1 \to +\infty$ and the subspace of solutions satisfying the boundary condition (1.11). Under our hypotheses, the Evans function may be defined to be C^∞ away from the origin on $\Re\lambda \geq 0$ with continous limits (typically depending on direction) at $(0, 0)$ along rays through the origin; see [Se2, SZ, GMWZ5, GMWZ6].

Definition 1.4. We define *strong spectral*, or *uniform Evans* stability as

$$|D(\tilde{\xi}, \lambda)| \geq \theta(C) > 0 \tag{D}$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$.

Remark 1.5. Under assumptions (H0)–(H6), uniform Evans stability implies linearized and nonlinear stability in both the long time and small viscosity limits of general noncharacteristic boundary layers (not necessarily associated with standing shocks) of symmetric hyperbolic–parabolic systems that are symmetric–dissipative at U_+ with Dirichlet boundary conditions $\dim \Upsilon_3 = 0$; see [GMWZ5, GMWZ6, NZ1, NZ2, N2]. For more general systems and boundary conditions, (D) augmented with a rescaled high-frequency condition has been shown in [GMWZ5, GMWZ6] to imply stability in the small viscosity limit.

1.2.2 The shock Evans function

Likewise, a necessary condition for linearized stability of the shock wave $\hat{U}$ is weak spectral stability, defined as nonexistence of unstable spectra $\Re\lambda > 0$ of the linearized operator L about the wave, or nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re\lambda > 0$ of the *shock Evans function*

$$\mathcal{D}(\tilde{\xi}, \lambda)$$

a Wronskian associated with (1.10) with columns consisting of bases of the subspace of solutions decaying as $x_1 \to +\infty$ and the subspace of solutions decaying as $x_1 \to -\infty$. Under our hypotheses, the shock Evans function may be defined to be C^∞ away from the origin on $\Re\lambda \geq 0$ and C^0 at the origin, with first directional derivatives (typically depending on direction) at $(0, 0)$ along rays through the origin; see [GZ, Z3, GMWZ4, GMWZ6].

Definition 1.6. Uniform Evans stability of a standing shock is defined as

$$|\mathcal{D}(\tilde{\xi}, \lambda)| \geq \theta(C) > 0 \tag{$\mathcal{D}$}$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$.

Remark 1.7. Under assumptions (H0)–(H7), uniform Evans stability ($\mathcal{D}$) implies linearized and nonlinear stability in both the long time and small viscosity limits of standing shocks of symmetric hyperbolic–parabolic systems that are symmetric–dissipative at $U_\pm$; see [GMWZ4, Z3, N2].

1.3 Main results

1.3.1 Convergence

Denote by $D_X(\tilde{\xi}, \lambda)$ the Evans function associated with $\bar{U}^X$ and $D_-(\tilde{\xi}, \lambda)$ the Evans function associated with the constant boundary-layer $U \equiv U_-$ at the lefthand endstate U_- of $\hat{U}$. Then, our first main result is as follows.

Theorem 1.8. *Under assumptions (H0)–(H7), there exists a continuous nonvanishing function $\beta(\tilde{\xi}, \lambda, X)$ such that*

$$D_X(\tilde{\xi}, \lambda) \to \beta(\tilde{\xi}, \lambda, X) D_-(\tilde{\xi}, \lambda) \mathcal{D}(\tilde{\xi}, \lambda) \tag{1.12}$$

as $X \to \infty$, uniformly on compact subsets of $\{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\}$.

Corollary 1.9. *Under (H0)–(H7), weak spectral stability of both the constant boundary-layer $U \equiv U_-$ and the standing shock $\hat{U}$ are necessary conditions for stability of $\bar{U}^X$ in the standing-shock limit $X \to \infty$.*

Remarks 1.10. 1. Under the stronger definition of uniform Evans stability defined in [GMWZ5, GMWZ6] involving also a rescaled high-frequency condition, and assuming (H0)–(H7), uniform Evans stability of the constant boundary-layer $U \equiv U_-$ and the standing shock $\hat{U}$ are also *sufficient* conditions for spectral stability of $\bar{U}^X$ in the standing-shock limit $X \to \infty$ for frequencies uniformly bounded away from the origin $(\tilde{\xi}, \lambda) = (0, 0)$. For intermediate frequencies $R^{-1} \leq |(\tilde{\xi}, \lambda)| \leq R$, this is an immediate consequence of Theorem 1.8. For high frequencies $|(\tilde{\xi}, \lambda)| \geq R$, $R > 0$ sufficiently large, it follows from the fact established in Section 3.2 [GMWZ5] that high-frequency stability is equivalent to stability of the constant layer $U \equiv U_0$, and the fact that $U_0^X \to U_-$ as $X \to \infty$. That is, *assuming stability of $\hat{U}$ and $U \equiv U_-$, unstable frequencies, if they occur, must converge to the origin as $X \to \infty$,* with no additional assumptions on the system (1.2).

2. It is shown in Corollary 1.29 [GMWZ5] that constant boundary-layers of symmetric–dissipative systems are uniformly Evans stable (under the stronger definition of [GMWZ5, GMWZ6]) for Dirichlet boundary conditions. Thus, *if* (1.2) *is symmetric–dissipative at U_- and the boundary conditions are Dirichlet-type, then the constant layer $U \equiv U_-$ is stable*, with the implications above. In the general case, stability or instability of the constant layer $U \equiv U_-$ may be determined by a linear algebraic computation, since the constant-coefficient eigenvalue ODE is explicitly soluble for each frequencies $(\tilde{\xi}, \lambda)$; see [GMWZ5] for further discussion.

1.3.2 One-dimensional stability

Due to translational invariance, $\mathcal{D}(0,0)=0$, and so we cannot conclude nonvanishing of D_X near the origin as $X\to\infty$ from the convergence result (1.12). Indeed, it is possible that the zero of $\mathcal{D}$ at the origin may perturb into the unstable half-plane $\Re\lambda>0$ for boundary layers with X large, yielding instability. In the one-dimensional setting, this may be dectected by the *stability index*

$$\Gamma := \operatorname{sgn} \lim_{\lambda\to 0^+ \text{real}} D(0,\lambda) \times \lim_{\lambda\to +\infty\, \text{real}} \operatorname{sgn} D(0,\lambda),$$

where D is chosen with a standard normalization guaranteeing that it is real for real λ and $\tilde\xi=0$. The stability index is well-defined by the properties that D is continuous along rays at the origin and nonvanishing for real λ sufficiently large; see [GZ, SZ, Z3] for further discussion. Negativity of Γ, by the Intermediate Value Theorem, implies existence of a real positive root $D(0,\lambda_*)=0$, hence *one-dimensional instability.*

Observing that stability of the constant layer $U\equiv U_-$ and the limiting shock $\hat U$ imply that $\operatorname{sgn} D_-(\lambda)$ and $\operatorname{sgn}\mathcal{D}(\lambda)$ are constant for real $\lambda\geq 0$, we thus obtain by convergence, (1.12), that a necessary condition for stability in the standing shock limit $X\to+\infty$, assuming stability of the constant layer, is nonnegativity of

$$\hat\Gamma := \lim_{X\to+\infty} \operatorname{sgn} \lim_{\lambda\to 0^+ \text{real}} \frac{D_X(0,\lambda)}{D_-(0,\lambda)} \times \lim_{\lambda\to 0^+ \text{real}} \operatorname{sgn}\mathcal{D}(0,\lambda), \tag{1.13}$$

provided these limits exist.

A standard shock stability result [GZ, ZS, Z3] is that, with appropriate normalization,

$$\begin{aligned} \lim_{\lambda\to 0^+ \text{real}} \operatorname{sgn}\mathcal{D}(0,\lambda) &= \operatorname{sgn}\partial_\lambda \mathcal{D}(0,0) = \operatorname{sgn}\delta, \\ \delta &:= \operatorname{sgn}\det(R^-, R^+, [U]), \end{aligned} \tag{1.14}$$

where R^- and R^+ are matrix blocks whose columns span the stable subspace of $dF_1(U_-)$ and the unstable subspace of $dF_1(U_+)$ and $[U]:=U_+-U_-$ denotes the jump across the shock. The determinant $\det(R^-,R^+,[U])$ may be recognized as the *Lopatinski determinant* of one-dimensional inviscid theory, whose nonvanishing is equivalent to one-dimensional inviscid shock stability.

Our second main result asserts that the first limit in (1.13) also exists, yielding a necessary stability condition of nonnegativity of a certain Lopatinski-like determinant $\hat\delta$ relative to the sign of δ.

Theorem 1.11. *Assuming (H0)–(H7) and stability of the constant-layer $U\equiv U_-$ and the limiting shock $\hat U$,*

$$\begin{aligned} \lim_{X\to+\infty} \operatorname{sgn} \lim_{\lambda\to 0^+ \text{real}} \frac{D_X(0,\lambda)}{D_-(0,\lambda)} &= \operatorname{sgn}\hat\delta, \\ \hat\delta &:= \operatorname{sgn}\det(\hat R^-, R_+, \hat V) \end{aligned}$$

provided $\hat\delta \neq 0$ *and* $S := \lim_{z\to-\infty} \frac{\hat U'}{|\hat U'|}(z)$ *exists, where* $\hat R^-$ *has the dimensions of* R^- *and* $\hat V$ *is a single column vector, hence*

$$\text{sgn}\delta\hat\delta = \text{sgn}\det(R^-, R_+, [U])\det(\hat R^-, R_+, \hat V) \geq 0 \tag{1.15}$$

is necessary for one-dimensional stability in the standing-shock shock limit.

For a Lax 1*-shock with Dirichlet boundary conditions,*

$$\delta = \det(R_+, [U]) \text{ and } \hat\delta = \det(R_+, dF_1(U_-)S) \tag{1.16}$$

so that (1.15) *becomes*

$$\text{sgn}\det(R_+, [U])\det(R_+, dF_1(U_-)S) \geq 0. \tag{1.17}$$

Remarks 1.12. 1. In the (characteristic) limit $U_- \to U_+$ as the amplitude of the background shock $\hat U$ goes to zero, $S \sim [U] \sim r_1^{\pm}$, where r_1 is the eigenvector of A_1 associated with the smallest eigenvalue a_1, and $dF_1(U_-)S \sim A_1^- r_1^- = a_1^-[U]$, where $a_1^- > 0$ by the Lax shock conditions (1.7). Thus, $\text{sgn}\det(R_+, [U])\det(R_+, dF_1(U_-)S) \sim \text{sgn}\det(R_+, [U])^2 > 0$, consistent with stability.

2. From the boundary-layer ODE, $dF_1(\hat U)\hat U' = (B_{11}(\hat U)\hat U')'$, giving $A_1(x_1)\hat U' = B_{11}\hat U''$, where $'$ denotes ∂_{x_1}. Thus, $A_1^- S = \alpha B_{11}^- S$, where $\alpha = \lim_{z\to-\infty}(\hat U'/\hat U)(z)$ is necessarily real and positive if the limit $S := \lim_{z\to-\infty}(\hat U'/|\hat U'|)(z)$ exists. It follows that we may replace (1.17) by the equivalent condition

$$\det(R_+, [U]))\det(R_+, B_{11}(U_-)S) \geq 0, \tag{1.18}$$

which may be recognized as the necessary condition derived by a rather different argument in Section 4.1 of [SZ].

Our third result states that for Lax 1-shocks the necessary conditions we have derived are also essentially *sufficient* for one-dimensional stability in the standing shock limit.

Theorem 1.13. *For a Lax* 1*-shock with general boundary conditions* (1.6)*, assuming (H0)–(H7), positivity of* $\hat\Gamma$ *together with stability of the constant layer* $U \equiv U_-$ *and the limiting shock* $\hat U$ *is* sufficient *for stability in the standing-shock limit* $X \to +\infty$*.*

Remarks 1.14. 1. It was shown in [HuZ] that in the limit $U_- \to U_+$, the background shock $\hat U$ is stable. For symmetric-dissipative systems with Dirichlet boundary conditions, the constant layer $U \equiv U_-$ is stable. By Remark 1.12.1, therefore, for Lax 1-shocks of symmetric-dissipative systems, with Dirichlet boundary conditions, layers $\hat U^X$ are one-dimensionally stable in the standing-shock limit $X \to +\infty$ for U_+ fixed and shock amplitude $|U_+ - U_-|$ sufficiently small.

2. By Remark 1.12.2, the necessary condition of [SZ], together with stability of the limiting shock $\hat U'$ and the constant layer $U \equiv U_-$, is also sufficient for one-dimensional stability. For ideal gas dynamics, the numerical study of [HLyZ1] indicates one-dimensional

stability of arbitrary shock waves $\hat{U}$ for gas constant γ within the physical range $1.2 \leq \gamma \leq 3$ (the only values considered); likewise, $U \equiv U_-$ is stable by symmetric–dissipativity of the compressible Navier–Stokes equations. Thus, *for ideal gas dynamics with Dirichlet boundary conditions and* $1.2 \leq \gamma \leq 3$, *one-dimensional stability in the standing Lax* 1*-shock limit is completely decided by the simple algebraic condition* (1.18) *of [SZ].*

For Lax p-shocks, $p \geq 2$, the situation is more complicated, involving a tricky double limit. In particular, the conditions of Theorem 1.11 are only necessary and not sufficient for stability.

1.3.3 Multi-dimensional stability

We restrict now to the case of a Lax 1-shock, for simplicity taking pure Dirichlet boundary conditions, $\dim \Upsilon_3 = 0$. In multi-dimensions, uniform inviscid stability of a Lax 1-shock is defined as nonvanishing of the multi-dimensional Lopatinski determinant

$$\Delta(\tilde{\xi}, \lambda) := \det \Big(\mathcal{R}_+(\tilde{\xi}, \lambda), \lambda[U] + \sum_{j=2}^{d} i\xi_j [F_j(U)] \Big), \tag{1.19}$$

on the nonnegative unit sphere $\mathcal{S}^+ := \{|(\tilde{\xi}, \lambda)| = 1, \, \Re\lambda \geq 0\}$, where $\mathcal{R}_+$ is a matrix blocks whose columns form a basis for the unstable subspaces of

$$\mathcal{A}_+(\tilde{\xi}, \lambda) := \Big(\lambda I + \sum_{j=2}^{d} i\xi_j dF_j(U_+)\Big) dF_1(U_+)^{-1} \tag{1.20}$$

and $[h(U)] := h(U_+) - h(U_-)$ denotes jump in h across the shock. Define the related determinant

$$\hat{\Delta}(\tilde{\xi}, \lambda, \eta) := \det \Big(\mathcal{R}_+(\tilde{\xi}, \lambda), \lambda[U] + \sum_{j=2}^{d} i\xi_j [F_j(U)] + \eta dF_1(U_-) S_- \Big), \tag{1.21}$$

$\eta \in \mathbb{R}$, where $S := \lim_{z \to -\infty} \frac{\hat{U}'}{|\hat{U}'|}(z)$. Then, our fourth and fifth main results, giving necessary conditions and sufficient conditions for multi-dimensional stability analogous to those of Theorems 1.11 and 1.13, are as follows.

Theorem 1.15. *For a Lax* 1*-shock and Dirichlet boundary conditions, assuming (H0)–(H7), a necessary condition for stability of* $\bar{U}_X$ *in the standing-shock limit* $X \to +\infty$ *is that* $\hat{\Delta}$ *have no root that is simple with respect to* λ *on the positive half-sphere*

$$\hat{\mathcal{S}}^+ := \{|\tilde{\xi}, \lambda, \eta| = 1, \, \Re\lambda > 0, \, \eta > 0\},$$

in the sense that $\hat{\Delta} = 0$ *and* $\partial_\lambda \hat{\Delta} \neq 0$.

Theorem 1.16. *For a Lax* 1*-shock and Dirichlet boundary conditions, assuming (H0)–(H7), sufficient conditions for stability of* $\bar U_X$ *in the standing-shock limit* $X \to +\infty$ *are nonvanishing of* $\hat\Delta$ *on the nonnegative half-sphere* $\hat{\mathcal{S}}^+ := \{|\tilde\xi, \lambda, \eta| = 1, \Re\lambda \ge 0, \eta \ge 0\}$*, stability of the limiting shock* $\hat U$*, and stability of the constant layer* $U \equiv U_-$*,*

Remark 1.17. In the one-dimensional case $\tilde\xi \equiv 0$, $\mathcal{R}_\pm \equiv R_\pm =$ constant, so that $\partial_\lambda \hat\Delta \equiv \delta$. It is readily seen that Theorems 1.15 and 1.16 reduce in this context to the restrictions of Theorems 1.11 and 1.13 to the case of a Lax 1-shock and Dirichlet boundary conditions, provided that the limiting shock $\hat U$ is one-dimensionally stable, so that $\delta \ne 0$ [ZH].

Remarks 1.18. 1. Multi-dimensional stability of shock waves in the limit $U_- \to U_+$ has been established in [FS] for symmetric dissipative systems with strictly parabolic (Laplacian) viscosity $\sum (B_{jk} U_{x_j})_{x_k}$ By the arguments of Remarks 1.12.1 and 1.14.1, therefore, boundary layers of such systems with Dirichlet boundary conditions are multi-dimensionally stable in the standing Lax 1-shock limit for U_+ fixed and $|U_+ - U_-|$ sufficiently small. The argument of [FS] appears likely to generalize to the general symmetric dissipative case (see also [PZ]), which would extend the boundary layer result also to the general case.

2. For ideal gas dynamics, the numerical study of [HLyZ2] indicates multi-dimensional stability of arbitrary shock waves for gas constants in the physical range $1.2 \le \gamma \le 3$. By the arguments of Remarks 1.12.2 and 1.14.2, therefore, *for ideal gas dynamics with Dirichlet boundary conditions and* $1.2 \le \gamma \le 3$*, multi-dimensional stability in the standing Lax* 1*-shock limit is completely decided by vanishing or nonvanishing of the extended Lopatinski determinant* $\hat\Delta$ *defined in* (1.21)*: a simple, linear-algebraic condition.*

1.3.4 Verification

Evidently, nonvanishing of $\hat\Delta$ on the nonnegative half-sphere in $(\tilde\xi, \lambda, \eta)$ is equivalent to the condition that the image of

$$\hat\eta(\tilde\xi, \lambda) := -\Delta(\tilde\xi, \lambda) / \det \Big(\mathcal{R}_+(\tilde\xi, \lambda), dF_1(U_-) S_- \Big) \tag{1.22}$$

over the nonnegative half-sphere in $(\tilde\xi, \lambda)$ avoid the nonnegative real axis, a natural generalization of the one-dimensional condition (1.17). Note that $\hat\eta$ is independent of the choice of $\mathcal{R}_+$ and homogeneous degree one in $(\tilde\xi, \lambda)$. This leads us to the following condition convenient for numerical or analytic verification.

Proposition 1.19. *Assuming the one-dimensional stability condition* (1.17)*, nonvanishing of* $\hat\Delta$ *on the nonnegative half-sphere in* $(\tilde\xi, \lambda, \eta)$ *is equivalent to the condition that the image of* $\hat\eta(1, i\tau)$ *over* $\tau \in \mathbb{R}$*, with* $\hat\eta$ *defined as in* (1.22)*, does not intersect the nonnegative real axis.*

Proof. By homogeneity of $\hat\Delta$, nonvanishing on the half-sphere is equivalent to nonvanishing of $\hat\delta(0, \lambda, \eta)$, or (1.17), and nonvanishing of $\hat\delta(1, \lambda, \eta)$, or the condition that $\hat\eta(1, \lambda)$ avoid the nonnegative real axis for $\Re\lambda \ge 0$. Recalling the standard fact that $\mathcal{R}_+(1, \lambda)$, hence $\hat\eta(1, \lambda)$, may be chosen to be analytic in λ for $\Re\lambda > 0$ and continuous at $\Re\lambda = 0$, we find by the argument principle applied to a sufficiently large semicircle about the origin, bounded to the left by the imaginary axis, and the fact that $\hat\eta(1,\lambda)$ by homogeneity/continuity does not intersect the nonnegative real axis for $\Re\lambda \geq 0$ and $|\lambda|$ sufficiently large (by the assumed one-dimensional stability) that the latter condition is equivalent to nonintersection with the nonnegative real axis for as λ traverses the imaginary axis. □

1.4 Discusssion and open problems

The results of Theorems 1.15 and 1.16 illuminate and greatly extend the earlier results of [SZ] and [CHNZ] in the one-dimensional case. In particular, we regard the derivation of necessary and sufficient conditions for multi-dimensional stability as a substantial advance. Though our necessary and our sufficient conditions are slightly different, the difference is sufficiently slight that it should not interfere in practice with classification of physical stability regions.

We note that, besides its independent interest, the treatment of the standing shock limit, as pointed out in [CHNZ], is important in truncating the computational domain for global stability analyses.

On the other hand, we have restricted here mainly to the simplest case of a Lax 1-shock, which corresponds to the case of *inflow* boundary conditions. It would be very interesting to obtain corresponding conditions also in the case of outflow boundary conditions, for example, a Lax n-shock. Likewise, it would be very interesting to carry out computations analogous to those carried out for gas dynamics in Section 5 also for the equations of MHD.

2 Construction of the Evans function

We begin by reviewing the construction of the Evans function following [Z3, GMWZ5, GMWZ6, NZ1, NZ2].

2.1 Expression as a first-order system

We first observe that matrix

$$\begin{pmatrix} A_1^{11} & A_1^{12} \\ B_{11}^{21} & B_{11}^{22} \end{pmatrix} = \begin{pmatrix} dF_1^{11} & dF_1^{12} \\ B_{11}^{21} & B_{11}^{22} \end{pmatrix}$$

is full rank, by (H2)–(H3) together with block structure assumption (1.4), as can be seen most easily by working in $\tilde W$-coordinates; see [Z1, Z3, MaZ3].

As a consequence, the Fourier-transformed eigenvalue equations (1.10), (1.11) may be written as a first-order system

$$\begin{aligned} Z' &= \mathbb{G}_X(\tilde\xi, \lambda, x_1)Z, \\ M(\tilde\xi)Z &= 0 \end{aligned} \tag{2.1}$$

in the convenient coordinates

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} := \begin{pmatrix} B_{11}^{21}u + B_{11}^{22}v \\ B_{11}U' - A_1U \end{pmatrix} = \begin{pmatrix} B_{11}^{21}u + B_{11}^{22}v \\ -A_1^{11}u - A_1^{12}v \\ B_{11}^{21}u' + B_{11}^{22}v' - A_1^{21}u - A_1^{22}v \end{pmatrix}; \tag{2.2}$$

see [Z3] for further discussion.

In the case of Dirichlet boundary conditions $\dim \Upsilon_3 = 0$, the boundary operator M is independent of $\tilde{\xi}$, with

$$\ker M = \{Z : \, Z_1 = 0, \, Z_2 = 0\} \tag{2.3}$$

in the inflow case and

$$\ker M = \{Z : \, Z_1 = 0\} \tag{2.4}$$

in the outflow case. In general, $\ker B(\tilde{\xi})$ depends on $\tilde{\xi}$ in a possibly complicated way. By the underlying relation $\bar{U}_X(x_1) = \hat{U}(x_1 - X)$, we have

$$\mathbb{G}_X(\tilde{\xi}, \lambda, x_1) = \mathbb{G}(\tilde{\xi}, \lambda, x_1 - X). \tag{2.5}$$

2.2 Conjugation to constant-coefficients

We next recall the following result established in [Z3, GMWZ5], a consequence of the conjugation lemma introduced in [MZ1] and the fact proved in [MaZ3, Z3] that $\hat{U}$ under hypotheses (H0)–(H7) converges exponentially to $U_\pm$ as $x_1 \to \pm\infty$. See, e.g., [Z3] for details.

Proposition 2.1. *There exist matrix-valued functions*

$$T_X^\pm(\tilde{\xi}, \lambda, x_1) = T^\pm(\tilde{\xi}, \lambda, x_1 - X),$$

uniformly bounded with bounded inverse for $x_1 \gtrless 0$, locally analytic in $(\tilde{\xi}, \lambda)$, such that

$$|(T^\pm - Id)(x_1)| \le Ce^{-\theta|x_1|} \text{ for } x_1 \gtrless 0, \tag{2.6}$$

$\theta > 0$, and $Z := T_X^\pm X$ satisfies the constant-coefficient equation

$$X' = \mathbb{G}_X^\pm(\tilde{\xi}, \lambda)X \text{ for } x_1 \gtrless X$$

$\mathbb{G}_X^\pm(\tilde{\xi}, \lambda) := \mathbb{G}_X^\pm(\tilde{\xi}, \lambda, \pm\infty)$, whenever Z satisfies (2.1).

2.3 Definition of the Evans function

Finally, we recall the following standard result, established in increasing generality in [SZ, Z1, Z3, GMWZ4, GMWZ6].

Proposition 2.2. *Under (H0)–(H7), there exist matrices* $\mathbb{E}^-$, $\mathbb{E}^+$, *and* $\mathbb{E}^0$ *whose columns form bases of the unstable subspace of* $\mathbb{G}^-$, *the stable subspace of* $\mathbb{G}^+$, *and* $\ker \mathcal{M}(\tilde\xi)$ *and matrices* $\mathbb{F}^-$ *and* $\mathbb{F}^+$ *whose columns form bases of the the stable subspace of* $\mathbb{G}^-$ *and the unstable subspace of* $\mathbb{G}^+$, C^∞ *on* $\{\Re\lambda \geq 0\} \setminus \{(0,0)\}$ *and continuously extendable along rays through the origin. Moreover,* $\mathbb{E}^+$ *and* $\mathbb{F}^+$ *and* $\mathbb{E}^-$ *and* $\mathbb{F}^-$ *are uniformly transverse on compact subsets of* $\{\Re\lambda \geq 0\}$ *and*

$$\dim \operatorname{Span} \mathbb{E}^- = \dim \operatorname{Span} \mathbb{E}^0 = (n+r) - \dim \operatorname{Span} \mathbb{E}^-. \tag{2.7}$$

Remark 2.3. In the one-dimensional setting, the subspaces of Proposition 2.2 may be defined as *globally analytic* functions on $\Re\lambda \geq -\eta$, $\eta > 0$, using a standard construction of Kato [Kat]; see, e.g., [GZ, Z3, CHNZ].

With these preparations, we define the shock Evans function precisely as the $(n+r) \times (n+r)$ Wronskian

$$\mathcal{D}(\tilde\xi, \lambda) := \det\Big(T^-\mathbb{E}^-, T^+\mathbb{E}^+\Big)|_{x_1=0} \tag{2.8}$$

and the boundary-layer Evans function as

$$D_X(\tilde\xi, \lambda) := \det\Big(\mathbb{E}^0, T_X^+\mathbb{E}^+\Big)|_{x_1=0} = \det\Big(\mathbb{E}^0, T^+\mathbb{E}^+\Big)|_{x_1=-X}. \tag{2.9}$$

The Evans function for the constant-layer $U \equiv U_-$ is given by

$$D_-(\tilde\xi, \lambda) := \det\Big(\mathbb{E}^0, \mathbb{F}^-\Big),$$

since in this case $\mathbb{E}^+ = \mathbb{F}^-$, or equivalently by

$$D_-(\tilde\xi, \lambda) := \det\Big(\tilde{\mathbb{E}}^{-*}\mathbb{E}^0\Big), \tag{2.10}$$

where

$$(\tilde{\mathbb{E}}^-, \tilde{\mathbb{F}}^-) = (\mathbb{E}^-, \mathbb{F}^-)^{-1*}$$

are dual bases to $(\mathbb{E}^-, \mathbb{F}^-)$ with respect to the standard complex inner product, M^* denoting adjoint, or conjugate transpose, of a matrix M.

2.4 Behavior near zero

For later use, we record the following refinement of Proposition 2.2 (also established in [SZ, Z1, Z3, GMWZ4, GMWZ6]), from which we may determine the behavior of $\mathcal{D}$, D_x as $(\tilde\xi, \lambda) \to (0,0)$.

Proposition 2.4. *Under (H0)–(H7), the limiting subspaces*

$$\lim_{\rho\to 0^+} \text{Span } T^+ e^{\mathbb{G}^+ x_1}\mathbb{E}^+(\rho\tilde{\xi}_0, \rho\lambda_0) \text{ and } \lim_{\rho\to 0^+} \text{Span } T^- e^{\mathbb{G}^- x_1}\mathbb{E}^-(\rho\tilde{\xi}_0, \rho\lambda_0)$$

$\rho \in \mathbb{R}$, $\Re\lambda_0 \geq 0$, *are spanned by the direct sum of fast modes*

$$Z_1 = b_{11}\phi, \quad (Z_2, Z_3) = 0,$$

with ϕ *satisfying* $B_{11}\phi' - A_1\phi = 0$ *and decaying at* $+\infty$ *[resp.* $-\infty$*] and slow modes*

$$Z_1 = *, \quad (Z_2, Z_3) = \mathcal{R},$$

with $\mathcal{R}$ *spanning the unstable [resp. stable] subspace of* $\mathcal{A}_+$ *[resp.* $\mathcal{A}_-$*, with*

$$\mathcal{A}_\pm(\tilde{\xi}, \lambda) := \Big(\lambda I + \sum_{j=2}^d i\xi_j dF_j(U_\pm)\Big) dF_1(U_\pm)^{-1},$$

expressible alternatively as $Z = T^\pm X$ *with*

$$X_1 = b_{11}(-A_1^\pm)^{-1}\mathcal{R}, \quad (X_2, X_3) = \mathcal{R} \tag{2.11}$$

constant. Symmetric decompositions hold for $T^+ e^{\mathbb{G}^+ x_1}\mathbb{F}^+$ *and* $T^- e^{\mathbb{G}^- x_1}\mathbb{F}^-$.

3 Basic convergence result

Proof of Theorem 1.8 for $(\tilde{\xi}, \lambda)$ *bounded away from the origin.* Viewing $\mathcal{D}$ and D_X as Wronskians of solutions of the same ODE (2.1), we may rewrite (2.9) using Abel's Theorem as

$$D_X(\tilde{\xi}, \lambda) := e^{\int_0^{-X} \text{Trace}\mathbb{G}(\tilde{\xi},\lambda,z)dz} \det\Big(\mathcal{S}^{-X\to 0}\mathbb{E}^0, T^+\mathbb{E}^+\Big)|_{x_1=0}, \tag{3.1}$$

where $\mathcal{S}^{y\to x}$ denotes the solution operator of (2.1).

Next, expand

$$\begin{aligned} \mathbb{E}^0 &= T^-(-X)T^-(-X)^{-1}\mathbb{E}^0 \\ &= T^-\mathbb{E}^0|_{x_1=-X} + O(e^{-\theta X}) \\ &= T^-\Pi_{\mathbb{E}^-}\mathbb{E}^0|_{x_1=-X} + T^-\Pi_{\mathbb{F}^-}\mathbb{E}^0|_{x_1=-X} + O(e^{-\theta X}), \end{aligned} \tag{3.2}$$

where

$$\Pi_{\mathbb{E}^-} = \mathbb{E}^-\tilde{\mathbb{E}}^{-*} \text{ and } \Pi_{\mathbb{F}^-} = \mathbb{F}^-\tilde{\mathbb{F}}^{-*} \tag{3.3}$$

denote the eigenprojections of $\mathbb{G}^-$ onto subspaces $\mathbb{E}^-$ and $\mathbb{F}^-$, noting that

$$T^-\Pi_{\mathbb{E}^-}\mathbb{E}^0 = (T^-\mathbb{E}^-)(\tilde{\mathbb{E}}^{-*}\mathbb{E}^0). \tag{3.4}$$

From (3.4), we obtain

$$
\begin{aligned}
&\det\Big(\mathcal{S}^{-X\to 0}T^-\Pi_{\mathbb{E}^-}\mathbb{E}^0, T^+\mathbb{E}^+\Big)|_{x_1=0}\\
(3.5)\qquad &= \det\Big(\mathcal{S}^{-X\to 0}T^-\mathbb{E}^-, T^+\mathbb{E}^+\Big)|_{x_1=0}\det(\tilde{\mathbb{E}}^{-*}\mathbb{E}^0)\\
&= \det\Big(\mathcal{S}^{-X\to 0}T^-\mathbb{E}^-, T^+\mathbb{E}^+\Big)|_{x_1=0}D_-.
\end{aligned}
$$

Next, expanding

$$\mathcal{S}^{y\to x}T^-(y) = T^-(x)e^{\mathbb{G}^-(x-y)}$$

and noting that

$$e^{\mathbb{G}^-(x-y)}\mathbb{E}^- = e^{\Pi_{\mathbb{E}^-}\mathbb{G}^-(x-y)}\Pi_{\mathbb{E}^-}\mathbb{E}^- = \mathbb{E}^-e^{\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^-(x-y)}, \tag{3.6}$$

where $\text{Trace}(\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^-) = \text{Trace}(\tilde{\mathbb{E}}^{-*}\mathbb{E}^-\mathbb{G}^-) = \text{Trace}\Pi_{\mathbb{E}^-}\mathbb{G}^-$, we find that

$$
\begin{aligned}
&\det\Big(\mathcal{S}^{-X\to 0}T^-\mathbb{E}^-, T^+\mathbb{E}^+\Big)|_{x_1=0}\\
(3.7)\qquad &= e^{\text{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}\det\Big(T^-\mathbb{E}^-, T^+\mathbb{E}^+\Big)|_{x_1=0}\\
&= e^{\text{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}\mathcal{D},
\end{aligned}
$$

where, since $\mathbb{E}^-$ is the unstable subspace of $\mathbb{G}^-$,

$$e^{\text{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}$$

is uniformly exponentially growing in X for $\Re\lambda \geq 0$. Indeed, for $(\tilde{\xi}, \lambda)$ bounded from the origin $\Re\lambda \geq 0$, each column of $\mathcal{S}^{-X\to 0}T^-\mathbb{E}^-$ is uniformly exponentially growing in X at rate at least

$$e^{\mu_* X},$$

where $\mu_*(\tilde{\xi}, \lambda)$ is the smallest real part of the (positive real part) eigenvalues of $\mathbb{G}^-$ associated with the unstable subspace $\mathbb{E}^-$.

By a similar argument, each column of $\mathcal{S}^{-X\to 0}T^-\Pi_{\mathbb{F}^-}\mathbb{E}^0$ is uniformly exponentially decaying in X, at rate $e^{\mu^* X}$, where μ^* is the largest real part of the (negative real part) eigenvalues of $\mathbb{G}^-$ associated with the stable subspace $\mathbb{F}^-$. Collecting information, we thus have

$$
\begin{aligned}
(3.8)\qquad D_X(\tilde{\xi}, \lambda) &= \beta(\tilde{\xi}, \lambda)\Big(D_X(\tilde{\xi}, \lambda)\mathcal{D}(\tilde{\xi}, \lambda) + O(e^{-\theta X}) + O(e^{(\mu_*-\mu^*)X})\Big)\\
&\to \beta(\tilde{\xi}, \lambda, X)D_X(\tilde{\xi}, \lambda)\mathcal{D}(\tilde{\xi}, \lambda)
\end{aligned}
$$

as $X \to +\infty$, exponentially in X, where

$$\beta(\tilde{\xi}, \lambda, X) := e^{\int_0^{-X}\text{Trace}\mathbb{G}(\tilde{\xi}, \lambda, z)dz}e^{\text{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}, \tag{3.9}$$

for $(\tilde{\xi}, \lambda)$ uniformly bounded away from the origin on $\Re\lambda \geq 0$. □

Remark 3.1. In general, the real parts of the eigenvalues of $\mathbb{E}^-$ and $\mathbb{F}^-$ can converge to zero as $(\tilde{\xi}, \lambda)$ approach the origin, so that μ_*, $\mu^* \to 0$ and the above convergence argument fails. In the special case of a Lax 1-shock, $\mu^* \to 0$, but μ_* remains strictly negative [Z3], and so we obtain uniform convergence by this argument on all of $\Re\lambda \geq 0$. Indeed, in the one-dimensional case, we obtain uniform convergence on $\Re\lambda \geq -\eta$, $\eta > 0$. (The only obstruction in the multi-dimensional case is that $\mathcal{D}$ is not defined on this set [Z3].)

Combining Remarks 1.10, 2.3, and 3.1, we obtain the following simple result reducing determination of one-dimensional stability to computation of the stability index.

Lemma 3.2. *Under (H0)–(H7) and stability of the constant layer $U \equiv U_-$, for a stable Lax 1-shock in one dimension, D_X has exactly one zero on $\Re\lambda > -\eta$, $\eta > 0$, for X sufficiently large, hence is stable if and only if its stability index is positive.*

Proof. By Remarks 1.10, under stability of $U \equiv U_-$, we may restrict attention to a compact set of frequencies, while by Remark 2.3 we may take D_X and $\mathcal{D}$ analytic on $\Re\lambda \geq -\eta$. As the uniform limit of analytic functions, we find that the number of zeros of $\mathcal{D}D_-$ on $\Re\lambda > -\eta$ is equal to the number of zeros of D_X for X sufficiently large. As the number of zeros of $\mathcal{D}$ is one and the number of zeros of D_- is zero, by stability, the number of zeros of D_X is one as asserted.

As the stability index Γ counts the parity of the number of nonstable roots $\Re\lambda > 0$ (see [GZ, Z3]), positivity of Γ corresponding to an even number of nonstable roots, and since $\Gamma = 0$ corresponds to instability [NZ1], we thus obtain stability if and only if $\Gamma > 0$. □

4 Behavior near the origin

Proof of Theorem 1.8 for $(\tilde{\xi}, \lambda) \to (0, 0)$. The shock evans function $\mathcal{D}$ is continuous at the origin, with $\mathcal{D}(0, 0) = 0$, and D_- is bounded on compact sets. Thus, to complete the proof of Theorem 1.8, it suffices to show that

$$D_X(\tilde{\xi}, \lambda)/\beta(\tilde{\xi}, \lambda) \to 0$$

as $(\tilde{\xi}, \lambda) \to 0$ and $X \to +\infty$, with β defined as in (3.9).

Noting that $T^-\mathbb{E}^-(0, 0)$ contains by continuity all exponentially decaying solutions of the one-dimensional eigenvalue equation, hence, in particular, $\hat{U}'(x_1)$, we may without loss of generality assign the value $\hat{U}'(x_1)$ to the first column of $T^-\mathbb{E}^-$ at $(0, 0)$. Moreover, noting that the strongly unstable subspace of $\mathbb{G}^-$, defined as the part whose eigenvalues have strictly positive real part even at $(0, 0)$, perturbs analytically, we may restrict the first column to this subspace, ensuring that the first column of $\mathcal{S}^{-X\to 0}T^-\mathbb{E}^-$ is analytic up to the origin and moreover grows exponentially in X, at rate $e^{\theta X}$, some $\theta > 0$, for $|(\tilde{\xi}, \lambda)|$ sufficiently small, hence contributions to D_X coming from the first columns of the second two terms in the last line of (3.2) are exponentially small as $X \to 0$ and can be ignored, while the contributions in other columns are at least bounded. Likewise, we may arrange that the first column of $T^+\mathbb{E}^+$ be analytic up to the origin and equal to $\hat{U}'$ at $(\tilde{\xi}, \lambda) = (0, 0)$.

In place of (3.8), therefore, we obtain the weaker estimate

$$
\begin{aligned}
|D_X(\tilde\xi,\lambda)|/\beta(\tilde\xi,\lambda) \le & \Big|D_X(\tilde\xi,\lambda)\det\Big(T^-(\mathbb{E}_1^-,O(1)),T^+\mathbb{E}^+\Big)_{x_1=0}\Big| \\
& + O(e^{-\theta X}) + O(e^{(\mu_*-\mu^*)X}) \\
& \to 0
\end{aligned}
\tag{4.1}
$$

as $(\tilde\xi,\lambda)\to(0,0)$, $X\to+\infty$, where $T^-\mathbb{E}_1^-$ and $T^+\mathbb{E}_1^+$ denote the first columns of $T^-\mathbb{E}^-$ and $T^+\mathbb{E}^+$, since by our choice of normalization $T^-\mathbb{E}_1^-$ and $T^+\mathbb{E}_1^+$ are continuous (indeed, analytic) at the origin and coincide for $(\tilde\xi,\lambda)=(0,0)$. This completes the proof of the theorem. □

Proof of Theorem 1.11. Fix $\tilde\xi\equiv 0$, so that $D(0,\lambda)$ is continuous in λ. Without loss of generality, take $D_-(0,0)=1$, with, moreover,

$$
\mathbb{E}^0(0,0)e^{-\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^-X} = T(-X)e^{-\mathbb{G}^-X}\Big(\mathbb{E}^-(0,0)+\mathbb{F}^-(0,0)\alpha\Big)
\tag{4.2}
$$

for some $k\times k$ matrix α, where $k=\dim\mathbb{F}^-$. Thus,

$$
\begin{aligned}
\mathcal{S}^{-X\to 0}\mathbb{E}^0 &= T^-(0)\Big(e^{\mathbb{G}^-X}T^-(-X)^{-1}\mathbb{E}^0\Big)\\
&= T^-(0)\Big(\mathbb{E}^-(0,0)+\mathbb{F}^-(0,0)\alpha\Big)e^{-\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^-X}.
\end{aligned}
\tag{4.3}
$$

Following the proof of Theorem 1.8, we find that the quantity

$$
D_X(0,0)/e^{\int_0^{-X}\mathrm{Trace}\mathbb{G}(0,0,z)dz}e^{\mathrm{Trace}\Pi_{\mathbb{E}^-}\mathbb{G}^-X}
$$

is given (exactly, with no exponentially decaying error) by

$$
\det\Big(T^-(\mathbb{E}^-+\mathbb{F}^-\alpha,T^+\mathbb{E}^+\Big)|_{x_1=0,\,(\tilde\xi,\lambda)=(0,0)},
$$

where $\mathrm{Trace}\mathbb{G}(0,0,z)$ and $\mathrm{Trace}\Pi_{EE^-}\mathbb{G}^-(0,0)$ are real, hence

$$
e^{\int_0^{-X}\mathrm{Trace}\mathbb{G}(0,0,z)dz} \text{ and } e^{\mathrm{Trace}\Pi_{\mathbb{E}^-}\mathbb{G}^-X}
$$

are real and positive.

Appealing to Proposition 2.4, we may arrange without loss of generality that the first k_1 columns of $T^-(0,0,x_1)e^{\mathbb{G}^-(0,0)x_1}\mathbb{E}^-(0,0)$ consist of functions $(b_{11}\phi_j,0)$, where ϕ_j, $j=1,\dots,k_1$ are solutions of

$$
B_{11}\phi_j' - A_1\phi_j = 0
\tag{4.4}
$$

that are uniformly exponentially decaying as $x_1\to-\infty$, hence exponentially growing in forward direction, and the remaining k_2 columns consist of functions $(*,r_j^-)$, $j=1,\dots,k_2$, where r_j^- are constant eigenvectors of A_1^- with negative eigenvalues a_j, and similarly for

$T^+\mathbb{E}^+$. Likewise, we may arrange that the columns of $T^-\mathbb{F}^-$ consist of l_1 solutions of (4.4) that are uniformly exponentially decaying in x_1 in forward direction and l_2 functions $(*, r_j^-)$, $j = 1, \dots, l_2$, where r_j^- are constant eigenvectors of A_1^- with positive eigenvalues a_j.

Finally, we may choose the first column of $T^-\mathbb{E}^-|_{x_1=0}$ as $\hat{U}'(0)$, noting that $\hat{U}'(0)$ lies also in $T^+\mathbb{E}^+|_{x_1=0}$. Combining these facts, we find that, up to an exponentially decaying error with respect to X, we may rewrite $D_X(0,0)/e^{\int_0^{-X} \mathrm{Trace}\mathbb{G}(0,0,z)dz} e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}$ as

$$\det \begin{pmatrix} * & \psi_2 & \cdots & \psi_{k_2} & * & * & \psi_{k_1+1} & \cdots & \psi_{n+1} \\ |\hat{U}'(-X)|\hat{V}_X & 0 & \cdots & 0 & \hat{\mathcal{R}}^- & \mathcal{R}^+ & 0 & \cdots & 0 \end{pmatrix} \tag{4.5}$$

evaluated at $x_1 = 0$, $(\tilde{\xi}, \lambda) = (0,0)$, where $\psi_j := b_{11}\phi_j$, $\begin{pmatrix} * \\ |\hat{U}'(-X)|\hat{V}_X \end{pmatrix}$ is the part of the first column of $\mathbb{F}^-\alpha$ involving only slow modes

$$((-a_j^-)^{-1} b_{11} r_j^-, r_j^-)$$

(recall (2.11)), and $\begin{pmatrix} * \\ \hat{\mathcal{R}}^- \end{pmatrix}$ is the part involving only slow modes of the block of $\mathbb{E}^- + \mathbb{F}^-\alpha$ corresponding to the slow block $\begin{pmatrix} * \\ \mathcal{R}^- \end{pmatrix}$ of $\mathbb{E}^-$. Referring to (4.2), we see that $\hat{V}_X$ has a limit $\hat{V}$ as $X \to +\infty$ so long as $S := \lim_{z\to -\infty}(\hat{U}'/|\hat{U}'|)(z)$ exists (the first column of $T^-(-X)e^{-\mathbb{G}^- X}\mathbb{E}^-$ being then approximately $|\hat{U}'(-X)|S$ as $X \to +\infty$, so that the slow component of the first column of $\mathbb{F}^-\alpha$ is approximately $|\hat{U}'(-X)|\hat{V}$ for a fixed $\hat{V}$ determined by S).

By a block determinant expansion, we have, therefore,

$$\begin{aligned} D_X(0,0)/e^{\int_0^{-X} \mathrm{Trace}\mathbb{G}(0,0,z)dz} e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X} &= \\ \sigma|\hat{U}'(-X)| \det(\psi_2, \dots, \psi_{n+1}) \det(\hat{\mathcal{R}}^-, \mathcal{R}^+, \hat{V}_X)|_{x_1=0,\, (\tilde{\xi},\lambda)=(0,0)} \\ \to \sigma|\hat{U}'(-X)| \det(\psi_2, \dots, \psi_{n+1}) \det(\hat{\mathcal{R}}^-, \mathcal{R}^+, \hat{V})|_{x_1=0,\, (\tilde{\xi},\lambda)=(0,0)}, \end{aligned}$$

where $\sigma = \pm 1$ depending on dimensions n, r and $\det(\psi_2, \dots, \psi_{n+1}) \neq 0$ assuming stability of the limiting shock $\hat{U}$ (else $\mathcal{D}$ would vanish to second instead of first order at the origin [Z3]). Normalizing

$$\mathrm{sgn}\sigma \det(\psi_2, \dots, \psi_{n+1}) = +1,$$

we obtain

$$\lim_{X\to+\infty} \mathrm{sgn} \lim_{\lambda\to 0^+ \text{ real}} D_X(0,\lambda) = \hat{\delta} := \det(\hat{R}^-, R_+, \hat{V})$$

as claimed.

Reviewing the computation in [Z3] of $\partial_\lambda \mathcal{D}(0,0)$, we find that this is the same normalization of ψ_j columns leading to the assumed normalization (1.14), whence (1.15) is necessary for stability by the discussion above the statement of the theorem.

Finally, for a Lax 1-shock, all eigenvalues of A_1^- are positive, hence the $\mathcal{R}^-$ block is empty in the computation above and the boundary conditions must be of inflow type. If also, the boundary conditions are Dirichlet type, then by (2.3), the first column of $\mathbb{E}^0(0,0)e^{-\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^- X}$ (since all of $\mathbb{E}^0$) is of form

$$\left(\mathbb{E}^0(0,0)e^{-\tilde{\mathbb{E}}^{-*}\mathbb{G}^-\mathbb{E}^- X}\right)_1 = \hat{\mathbb{E}}_1^- + \left(\hat{\mathbb{F}}^{-,slow}\alpha\right)_1 = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ * \end{pmatrix} \end{pmatrix},$$

where $\hat{\mathbb{E}}_1^-$ and $\hat{\mathbb{F}}_1^{-,slow}$, defined as the first column of $\hat{\mathbb{E}}^- := T^-(-X)e^{-\mathbb{G}^- X}\mathbb{E}^-$ and the slow part of $\hat{\mathbb{F}}^- := T^-(-X)e^{-\mathbb{G}^- X}\mathbb{F}^-$, are asymptotically of form (since $T^- e^{\mathbb{G}^- x_1}\mathbb{E}_1^- = (b_{11}\hat{U}'(x_1), 0)$ and $\lim_{z\to-\infty} T^- = Id$)

$$|\hat{U}'(-X)| \begin{pmatrix} b_{11}^- S \\ 0 \end{pmatrix} \text{ and } |\hat{U}'(-X)| \begin{pmatrix} b_{11}^-(-A_1^-)^{-1}\hat{\mathcal{R}}_1^- \\ \hat{\mathcal{R}}_1^- \end{pmatrix}$$

as $X \to +\infty$, where $\hat{\mathcal{R}}_1^-$ lies in the unstable subspace of A_1^-. Equating, we find that $\hat{\mathcal{R}}_1^- = \begin{pmatrix} 0 \\ c \end{pmatrix}$ and $b_{11}^- S = b_{11}^-(A_1^-)^{-1}\hat{\mathcal{R}}_1^-$. Recalling that $\left(A_1^{11}, A_1^{12}\right)\hat{U}' = 0$ by the linearized boundary-layer ODE, so that

$$\left((A_1^-)^{11}, (A_1^-)^{12}\right)S = 0,$$

we find by inspection that $\hat{\mathcal{R}}_1 = A_1^- S$, or, in the notation of the general case,

$$\hat{V} = A_1^- S, \tag{4.6}$$

yielding the result. □

Proof of Theorem 1.13. Immediate, by Lemma 3.2 and Theorem 1.11. □

Proof of Theorem 1.15. Set $\eta := |\hat{U}'(-X)|$, and consider $0 \le \rho \le Ce^{-\theta X}$ for some fixed $C > 0$ and $\theta > 0$ sufficiently small, in particular, small enough that

$$|\hat{U}'(-X)| << Ce^{-\theta X}, \tag{4.7}$$

setting

$$(\tilde{\xi}, \lambda, \eta) =: (\rho\tilde{\xi}_0, \rho\tilde{\lambda}_0, \rho\eta_0)$$

with $|(\tilde{\xi}_0, \tilde{\lambda}_0, \eta_0)| = 1$. Note, as $X \to +\infty$, that the set of possible values of $(\tilde{\xi}_0, \tilde{\lambda}_0, \eta_0)$ expands to the full positive half-sphere $\Re\lambda_0$, $\eta_0 > 0$. Restrict now to a compact subset of the positive half-sphere, recalling (see Proposition 2.2) that the Evans functions D_X and $\mathcal{D}$, and their component columns, are C^∞ in ρ, $\tilde{\xi}_0$, $\tilde{\lambda}_0$.

Within the specified parameter-regime, both slow and fast modes of (2.1) at $(\tilde\xi,\lambda)=\rho(\tilde\xi_0,\lambda_0)$ are well-approximated on $x_1\in[-X,0]$ by their limiting values as $\rho\to 0$, described in Proposition 2.4. Mimicking the one-dimensional computations (4.5), (4.6), we may rewrite

$$D_X(\rho\tilde\xi_0,\rho\lambda_0)/e^{\int_0^{-X}\mathrm{Trace}\mathbb{G}(\rho\tilde\xi_0,\rho\lambda_0,z)dz}e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X}$$

as the sum of

$$\rho\det\begin{pmatrix} * & \psi_2 & \cdots & \psi_{k_2} & * & * & \psi_{k_1+1} & \cdots & \psi_{n+1}\\ \eta_0 A_1 S & 0 & \cdots & 0 & \hat{\mathcal{R}}^- & \mathcal{R}^+ & 0 & \cdots & 0\end{pmatrix} + o(\rho\eta_0)$$

and

$$\rho\det\begin{pmatrix} * & \psi_2 & \cdots & \psi_{k_2} & * & * & \psi_{k_1+1} & \cdots & \psi_{n+1}\\ \partial_\rho T^-\mathbb{E}_1^- - \partial_\rho T^+\mathbb{E}_1^+ & 0 & \cdots & 0 & \hat{\mathcal{R}}^- & \mathcal{R}^+ & 0 & \cdots & 0\end{pmatrix} + o(\rho|(\tilde\xi_0,\lambda_0)|),$$

both evaluated at $x_1=0$, $\rho=0$, and $(\tilde\xi_0,\lambda_0)$. We omit the details of this straightforward but tedious computation.

A standard computation [ZS, Z3, GMWZ4] using the variational equations of (2.1) with respect to ρ yields

$$\partial_\rho T^-\mathbb{E}_1^- - \partial_\rho T^+\mathbb{E}_1^+ = \lambda_0[U]+\sum_{j=2}^d i\tilde\xi_0^j[F_j(U)],$$

whence, normalizing as usual so that

$$\mathrm{sgn}\sigma\det(\psi_2,\dots,\psi_{n+1})=+1,$$

we obtain by block determinant expansion

$$\rho^{-1}D_X(\rho\tilde\xi_0,\rho\lambda_0)/e^{\int_0^{-X}\mathrm{Trace}\mathbb{G}(\rho\tilde\xi_0,\rho\lambda_0,z)dz}e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X} = \hat\Delta(\tilde\xi_0,\lambda_0,\eta_0)+o(1), \tag{4.8}$$

where $\hat\Delta$ is defined as in (1.21) and $o(1)$ is C^1 with respect to $\tilde\xi_0$, λ_0, and η_0 for each fixed X and $\to 0$ uniformly as $X\to 0$.

By an application of the Implicit Function Theorem, it follows that existence of a root $(\tilde\xi_0^*,\lambda_0^*,\eta_0^*)$ of $\hat\Delta$ on $\Re\lambda_0>0$ at which $\partial_{\lambda_0}\hat\Delta\neq 0$ implies existence of a nearby root $(\tilde\xi_0^*,\lambda_0^\dagger,\tilde\eta_0^*)$, $\Re\lambda_0^\dagger>0$, $\rho>0$, of

$$\rho^{-1}D_X(\rho\tilde\xi_0,\rho\lambda_0)/e^{\int_0^{-X}\mathrm{Trace}\mathbb{G}(\rho\tilde\xi_0,\rho\lambda_0,z)dz}e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X},$$

hence of D_X, for X sufficiently large, or instability of $\bar U^X$. Thus, nonvanishing of $\hat\Delta$ on the strictly positive half-sphere is necessary for stability as $X\to 0$. $\square$

Proof of Theorem 1.16. The estimate (4.8) in fact holds for all $\eta := |\hat{U}'(-X)|$ and $0 \le \rho \le Ce^{-\theta X}$, with the $o(1)$ term uniformly decaying and uniformly C^0 as $X \to \infty$ (however, not uniformly C^1; see [GMWZ4, GMWZ5, GMWZ6]). It follows therefore, that nonvanishing of $\hat{\Delta}$ on the (closed) nonnegative half-sphere, implying a lower bound on $|\hat{\Delta}|$, implies nonvanishing of D_X on the parameter range $0 \le \rho \le Ce^{-\theta X}$, for X sufficiently large.

If $Ce^{-\theta X} \le \rho << 1$, on the other hand, a much cruder estimate yields

$$\begin{aligned}\rho^{-1} D_X(\rho\tilde{\xi}_0, \rho\lambda_0)/e^{\int_0^{-X} \mathrm{Trace}\mathbb{G}(\rho\tilde{\xi}_0,\rho\lambda_0,z)dz} e^{\mathrm{Trace}(\Pi_{\mathbb{E}^-}\mathbb{G}^-)X} =& \\ \Delta(\tilde{\xi}_0, \tilde{\lambda}_0) + o(1) + O(|\hat{U}'(-X)|/Ce^{-\theta X}) =& \\ \Delta(\tilde{\xi}_0, \tilde{\lambda}_0) + o(1),&\end{aligned}$$

by (4.7), again with $o(1)$ uniformly decaying as $X \to +\infty$. This implies nonvanishing of D_X on the parameter range $1 >> \rho \ge Ce^{-\theta X}$, for X sufficiently large.

For ρ bounded from below, on the other hand, we have by the basic convergence result of Theorem 1.8 that D_X is nonvanishing if $\mathcal{D}$ and D_- are nonvanishing, i.e., if $\hat{U}$ and $U \equiv U_-$ are stable. This completes the proof of the theorem. □

5 Application to gas dynamics

We now apply our results to the fundamental example of compressible gas dynamics, restricting without loss of generality (by rotational invariance of the equations) to dimension $d = 2$. Consider the compressible Navier–Stokes equations

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \tag{5.1a}$$

$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy}, \tag{5.1b}$$

$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx}, \tag{5.1c}$$

$$\begin{aligned}(\rho E)_t + (\rho uE + up)_x + (\rho vE + vp)_y \\ = \Big(\kappa T_x + (2\mu + \eta)uu_x + \mu v(v_x + u_y) + \eta uv_y\Big)_x \\ + \Big(\kappa T_y + (2\mu + \eta)vv_y + \mu u(v_x + u_y) + \eta vu_x\Big)_y\end{aligned} \tag{5.1d}$$

on the half-plane $x \in \mathbb{R}^+$, $y \in \mathbb{R}$, where ρ is density, u and v are velocities in x and y directions, p is pressure,

$$E = e + \frac{u^2}{2} + \frac{v^2}{2}. \tag{5.2}$$

is total energy density, e and T are internal energy density and temperature, and constants $\mu > |\eta| \geq 0$ and $\kappa > 0$ are coefficients of first ("dynamic") and second viscosity and heat conductivity.

We assume ideal ("γ-law") gas equations of state

$$p_0(\rho, T) = \Gamma \rho e, \quad e_0(\rho, T) = c_v T, \tag{5.3}$$

where $c_v > 0$ is the specific heat at constant volume, $\Gamma := \gamma - 1 > 0$, and $\gamma > 1$ is the adiabatic index of the gas; equivalently,

$$p(v, S) = a e^{S/c_v} \rho^\gamma, \tag{5.4}$$

where S is thermodynamical entropy [Ba, Sm]. In the notation of (1.2), we have

$$\tilde{U} = (\rho, \rho u, \rho v, \rho E) =: (\rho, m_1, m_2, \mathcal{E}) \tag{5.5}$$

and

$$F_1(\tilde{U}) = (\rho u, \rho u^2 + p, \rho v^2, \rho u E) = (m_1, m_1^2/\rho + p, m_1 m_2/\rho, m_1 \mathcal{E}/\rho). \tag{5.6}$$

Remark 5.1. In the thermodynamical rarified gas approximation,

$$\gamma = \frac{2n+3}{2n+1}, \qquad \nu/\mu = \frac{9\gamma - 5}{4} \qquad \eta = -\frac{2}{3}\mu \tag{5.7}$$

for $\nu := \kappa/c_v$, where n is the number of constituent atoms of gas molecules (here assumed to have "tree" structure) [Ba], with $\gamma = 5/3$ and $\gamma = 7/5$ for the main applications of monatomic and diatomic gas. In particular,

$$1 < \gamma < 2 \quad \text{and} \quad \frac{\nu}{2\mu + \eta} > 1 \tag{5.8}$$

for common gases, a conclusion that is born out by experiment. See Appendices A and B of [HLyZ1] for further discussion.

5.1 Viscous Shock Profiles

From (5.1), setting time-derivatives to zero, integrating in x, and rearranging, we obtain after a brief calculation the standing-shock ODE

$$\begin{aligned} \hat{u}' &= (2\mu + \eta)^{-1}\Big(m(\hat{u} - u_-) + \Gamma(\hat{\rho}\hat{e} - \hat{\rho}_- e_-)\Big), \\ \hat{e}' &= \nu^{-1}\Big(m\big(\hat{e} - e_-\big) - \frac{m(\hat{u} - u_-)^2}{2} + (\hat{u} - u_-)\Gamma \hat{\rho}_- T_-\Big), \end{aligned} \tag{5.9}$$

where $m := \hat{\rho}\hat{u} \equiv$ constant and $\hat{v} \equiv$ constant.

Using various scale-invariances of system (5.1), we may take without loss of generality $m = \rho_- = u_- = 1$, $v_- = v_+ = 0$, yielding

$$\begin{aligned} \hat{u}' &= \frac{1}{2\mu+\eta}\left((\hat{u}-1) + \Gamma\left(\frac{\hat{e}}{\hat{u}} - e_-\right)\right), \\ \hat{e}' &= \nu^{-1}\left((\hat{e}-e_-) - \frac{(\hat{u}-1)^2}{2} + (\hat{u}-1)\Gamma e_-\right) \end{aligned} \tag{5.10}$$

with $\hat{v} \equiv 0$, with endstates

$$e_+ = \frac{u_+\alpha(u_+-1)}{\Gamma(\Gamma+2-\alpha)}, \quad e_- = \frac{(u_+-1)(\Gamma+2)}{\Gamma(\Gamma+2-\alpha)}, \quad \rho_+ = 1/u_+, \tag{5.11}$$

$\alpha := \frac{\Gamma+2-\Gamma u_+}{u_+-u_*}$, parametrized by the single quantity

$$1 \geq u_+ > u_* := \frac{\Gamma}{\Gamma+2}. \tag{5.12}$$

In the strong-shock limit $u_+ \to u_*$, $e_- \to 0$, with all other quantities remaining in physical range; for details of these computations, see [HLyZ2], Sections 3–5.

Linearizing (5.10) about $(u_-, e_-) = (1, e_-)$, we obtain

$$\begin{pmatrix} u \\ e \end{pmatrix}' = M_- \begin{pmatrix} u \\ e \end{pmatrix}, \qquad M_- := \begin{pmatrix} \frac{1}{2\mu+\eta} & 0 \\ 0 & \frac{1}{\nu} \end{pmatrix} \begin{pmatrix} 1-\Gamma e_- & \Gamma \\ \Gamma e_- & 1 \end{pmatrix}, \tag{5.13}$$

determining the asymptotic behavior of $(\hat{u}, \hat{e})(z)$ as $z \to -\infty$. One may check for all $1 \geq u_+ > u_*$ that M_- has two positive distinct real eigenvalues $0 < \omega_- \leq 1/\nu \leq \omega_+$,

$$\omega_\pm = \frac{1}{\nu} + \frac{\left(\frac{1-\Gamma e_-}{2\mu+\eta} - \frac{1}{\nu}\right) \pm \sqrt{\left(\frac{1-\Gamma e_-}{2\mu+\eta} - \frac{1}{\nu}\right)^2 + \frac{4\Gamma e_-}{(2\mu+\eta)\nu}}}{2},$$

with associated eigenvectors $s_j = (-1, -\frac{\Gamma e_-}{\nu(\omega_j - 1/\nu)})^T$, merging in the special limiting case $u_+ \to u_*/e_- \to 0$, $2\mu + \eta = \nu$ to a pair of real semisimple eigenvalues.

That is, for a Lax 1-shock, U_- is a repellor for the standing-wave ODE, and U_+ a saddle, in agreement with the abstract conclusions of [MaZ3] for extreme shocks of general systems and of [Gi] for shock profiles of gas dynamics with general equation of state. In particular, note that

$$\det M_- = (\nu(2\mu+\eta))^{-1}(1 - \Gamma(1+\Gamma)e_-) > 0,$$

with $(1 - \Gamma(1+\Gamma)e_-)$ approaching 1 in the strong shock limit $u \to u_*/e_- \to 0$, and $1 - \frac{2(\Gamma+1)}{(\Gamma+2)^2} > \frac{\Gamma^2+2}{(\Gamma+2)^2}$ in the weak shock limit $u_+ \to 1$.[1] By reality and simplicity of the eigenvalues ω_j, we have that limits

$$s := \lim_{z \to -\infty} (\hat{u}', \hat{e}')/|(\hat{u}', \hat{e}')| \tag{5.14}$$

[1] This repairs an error of [SZ], in which U_- was mistakenly computed to be a saddle, leading to an incorrect value of S.

and

$$S := \lim_{z\to-\infty} (\hat{U}'/|\hat{U}'|) = \frac{\partial U}{\partial(u,e)}|_{U_-} s = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/2 - e_- & 1 \end{pmatrix} s \tag{5.15}$$

exist, with s generically lying parallel to the slow mode s_-, or

$$\begin{aligned} S &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/2 - e_- & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -\frac{\Gamma e_-}{\nu(\omega_- - 1/\nu)} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ e_- \frac{\Gamma - \nu(\omega_- - 1/\nu)}{\nu(\omega_- - 1/\nu)} - \frac{1}{2} \end{pmatrix}. \end{aligned} \tag{5.16}$$

Finally, from (5.5)–(5.6), we obtain after a brief calculation

$$\begin{aligned} A_1^- := \partial(F_1/\partial\tilde{U})(U_-) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ p_\rho - 1 + p_e/2 & 2 - p_e & 0 & p_e \\ 0 & 0 & 1 & 0 \\ -1/2 & 1/2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Gamma e_- - 1 + \Gamma/2 & 2 - \Gamma/2 & 0 & \Gamma \\ 0 & 0 & 1 & 0 \\ -1/2 & 1/2 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.17}$$

from which we compute

$$A_1^- S = \begin{pmatrix} 0 \\ \Gamma e_- - 1 + \Gamma/2 + \Gamma e_- \frac{\Gamma - \nu(\omega_- - 1/\nu)}{\nu(\omega_- - \Gamma/\nu)} \\ 0 \\ e_- \frac{\Gamma - \nu(\omega_- - 1/\nu)}{\nu(\omega_- - 1/\nu)} - 1 \end{pmatrix}. \tag{5.18}$$

$$[U] = \Big(\frac{1-u_+}{u_+}, 0, 0, \frac{1-u_+}{2}\Big)^T. \tag{5.19}$$

5.1.1 The strong shock limit

For $\frac{\nu}{2\mu+\eta} < 1$, (5.16) converges to $S = (1, 0, 0, -1/2)^T$ in the strong shock limit $e_- \to 0$. For $\frac{\nu}{2\mu+\eta} \geq 1$, however, (5.16) becomes singular in the limit as $e_- \to 0$, for which also

$\omega_- \to 1/\nu$. To evaluate this limit, it is easier to return to (5.13) and compute directly with $e_- = 0$, to obtain $s \to (-1, 1-\phi)^T$, yielding the general formula

$$S \to (1,0,0,1/2 - \min\{1,\phi\})^T \text{ as } u_+ \to u_-, \quad \phi := \frac{2\mu+\eta}{\nu}. \tag{5.20}$$

Noting that

$$A_1 - \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1+\Gamma/2 & 2-\Gamma/2 & 0 & \Gamma \\ 0 & 0 & 1 & 0 \\ -1/2 & 1/2 & 0 & 1 \end{pmatrix}, \tag{5.21}$$

we thus have

$$A_1 S - \to \begin{pmatrix} 0 \\ \Gamma \max\{0, 1-\phi\} - 1 \\ 0 \\ -\min\{1,\phi\} \end{pmatrix} \tag{5.22}$$

and

$$[U] \to \Big(\frac{2}{\Gamma}, 0, 0, \frac{1}{\Gamma+2}\Big)^T, \tag{5.23}$$

completing our asymptotic analysis.

5.2 One-dimensional stability

Following the treatment in [Se1, SZ], we note for Lax 1-shocks that

$$\det(\mathcal{R}_+, f) = \ell_+ \cdot f, \tag{5.24}$$

for any vector $f \in \mathbb{C}^n$, where ℓ_+ is the unique stable left eigenvector of $\mathcal{A}_+$ and $\cdot$ denotes complex inner product. In one dimension, ℓ_+ is just the stable left eigenvector of A_+, which may be computed to be

$$\begin{aligned} \ell_+ &= \Big(p_\rho + cu + \frac{p_e(u^2/2 - e)}{\rho}, -\frac{p_e u}{\rho} - c, 0, \frac{p_e}{\rho}\Big)^T (U_+) \\ &= \Big(c_+ u_+ + \Gamma u_+^2/2, -\Gamma u_+ - c_+, 0, \Gamma\Big)^T, \end{aligned} \tag{5.25}$$

where

$$c := \sqrt{p p_e/\rho^2 + p_\rho} = \sqrt{\Gamma(\Gamma+1)e} \tag{5.26}$$

denotes sound speed. This computation is most easily accomplished by working in the more convenient nonconservative coordinates (ρ, u, v, e), which are related to conservative variables $(\rho, \rho u, \rho v, \rho(e + u^2/2 + v^2/2))$ by a readily computed lower triangular change of coordinates; see [Se1] or Appendix A.

Combining all facts, we have

$$\begin{aligned}\hat{\delta} &= \ell_+ \cdot A_- S \\ &= \Big(-\Gamma u_+ - c_+ \Big)\Big(\Gamma e_- - 1 + \Gamma/2 + \Gamma e_- \frac{\Gamma - \nu(\omega_- - 1/\nu)}{\nu(\omega_- - \Gamma/\nu)}\Big) \\ &\quad + \Gamma \Big(e_- \frac{\Gamma - \nu(\omega_- - 1/\nu)}{\nu(\omega_- - 1/\nu)} - 1 \Big).\end{aligned} \tag{5.27}$$

It is readily verified on the other hand that

$$\delta > 0; \tag{5.28}$$

see Section 5.2.1 just below. The one-dimensional stability condition (1.17) thus reduces in this case to

$$\mathrm{sgn} \hat{\delta} > 0, \tag{5.29}$$

a condition that can be readily checked numerically using (5.27).

5.2.1 The strong shock limit

In the strong shock limit $u_+ \to u_*$, we have $e_- \to 0$, $\alpha \to +\infty$, and

$$e_+ \to u_*(1 - u_*)/\Gamma = 2/(\Gamma + 2)^2,$$

so that $c_+ \to \sqrt{2\Gamma(\Gamma + 1)}/(\Gamma + 2)$ and

$$\ell_+ \to \Big(\frac{\Gamma\sqrt{2\Gamma(\Gamma+1)}}{(\Gamma+2)^2} + \frac{\Gamma^3}{2(\Gamma+2)^2}, -\frac{\Gamma^2}{\Gamma+2} - \frac{\Gamma\sqrt{2\Gamma(\Gamma+1)}}{\Gamma+2}, 0, \Gamma \Big)^T,$$

hence, by (5.22),

$$\begin{aligned}\hat{\delta} &= \ell_+ \cdot A_- S \\ &\to \Big(-\frac{\Gamma^2}{\Gamma+2} - \frac{\Gamma\sqrt{2\Gamma(\Gamma+1)}}{\Gamma+2} \Big)\Big(\Gamma \max\{0, 1 - \phi\} - 1\Big) - \Gamma \min\{1, \phi\}.\end{aligned} \tag{5.30}$$

Meanwhile,

$$\delta = \ell_+ \cdot [U] \to \frac{2\sqrt{2\Gamma(\Gamma+1)}}{(\Gamma+2)^2} + \frac{\Gamma}{(\Gamma+2)^2} + \frac{\Gamma}{\Gamma+2} > 0,$$

from which we may conclude by homotopy/nonvanishing of δ that $\delta > 0$ for all $1 \geq u_+ \geq u_*$, verifying (5.28)–(5.29).

The case $\phi \geq 1$. For $\phi \geq 1$, (5.29) becomes

$$\Big(\frac{\Gamma\sqrt{2\Gamma(\Gamma+1)}}{\Gamma+2} + \frac{\Gamma^2}{\Gamma+2}\Big) - \Gamma > 0,$$

or

$$\sqrt{2\Gamma(\Gamma+1)} > 2,$$

which evidently fails for Γ in the kinetic range $0 \leq \Gamma \leq 1$ (indeed, for all Γ outside $(1,2)$). Thus, we may conclude instability in the strong shock limit in this range.

The case $\phi \leq 1$. For $\phi \leq 1$, (5.29) becomes

$$\Big(-\frac{\Gamma^2}{\Gamma+2} - \frac{\Gamma\sqrt{2\Gamma(\Gamma+1)}}{\Gamma+2}\Big)\Big(\Gamma(1-\phi)-1\Big) - \Gamma\phi > 0,$$

or

$$\Big(\sqrt{2\Gamma(\Gamma+1)}+\Gamma\Big)\Big(1-\Gamma(1-\phi)\Big) - (\Gamma+2)\phi > 0.$$

Defining $\sigma := \sqrt{2\Gamma(\Gamma+1)}+\Gamma$, we may rewrite this as

$$\sigma(1-\Gamma) > (\Gamma - \Gamma\sigma + 2)\phi,$$

or, assuming $\Gamma(1-\sigma)+2 > 0$, as holds for example on the kinetic range $0 < \Gamma < 1$, or $1 < \gamma < 2$ (on which $\sigma < 2+\Gamma$, so $2+\Gamma > \Gamma\sigma$), as

$$\phi < \frac{\sigma(1-\Gamma)}{\Gamma(1-\sigma)+2}, \tag{5.31}$$

which is satisfied for ϕ small enough, but for $\phi = 1$, hence for $\phi \leq 1$ large enough, is not satisfied, by the analysis of case $\phi = 1$ above.

Common gases and the kinetic approximation. Recall that for common gases, ϕ is less than one. For gases obeying the kinetic approximation (5.7)–(5.8),

$$\phi = \frac{16}{27\Gamma + 12}, \tag{5.32}$$

so that $\phi < 1$ for $\Gamma \geq 4/27 \approx .148$, in particular for n-atomic gases with $n \leq 5$. Thus, it is the case $\phi \leq 1$ that is relevant to typical applications. Substituting (5.32) into (5.31) and noting that $\sqrt{2\Gamma(\Gamma+1)} \leq \Gamma + 1$ for $0 < \Gamma < 1$ yields the necessary condition

$$16(\Gamma+2) < (2\Gamma+1)(1+15\Gamma), \tag{5.33}$$

or $0 < (\Gamma - 1)(30\Gamma + 31)$, which is violated for the entire kinetic range $0 < \Gamma < 1$.

Conclusions By Remarks 1.12.2 and 1.18.2, boundary layers are both one- and multi-dimensionally stable in the standing shock limit for limiting shocks of sufficiently small amplitude, i.e., $1 - u_+$ sufficiently small. By the calculations above, however, for typical gas laws, they are not even one-dimensionally stable in the strong shock limit for limiting shocks of sufficiently large amplitude, i.e., $u_+ - u_*$ sufficiently small, even though the corresponding shock is perfectly stable [HLyZ1, HLyZ2].

Thus, we have the striking conclusion that *for (all!) typically physically occurring gases under inflow Dirichlet boundary conditions, there is a transition from stability to instability of boundary layers in the standing shock limit as the amplitude of the limiting shock increases from zero to its maximum value.*

5.3 Multi-dimensional stability

The computation of $\ell_+(\tilde\xi,\lambda)$ in multi-dimensions may be found, for example, in Appendix C, [Z3][2], where it is computed as

$$\ell_+(\tilde\xi,\lambda) = \Big(\theta - \frac{ic\beta u}{\sqrt{\tilde\xi^2-\beta^2}} + \frac{\eta u^2}{\beta}, \frac{ic\beta}{\sqrt{\tilde\xi^2-\beta^2}} - \eta u, \frac{c\tilde\xi}{\sqrt{\tilde\xi^2-\beta^2}}, \eta\Big)_+, \tag{5.34}$$

where $\theta := p_\rho - \frac{p_e e}{\rho} = 2\Gamma e$, $\eta := \frac{p_e}{\rho} = \Gamma$, and c is sound speed (5.26), or

$$\ell_+(\tilde\xi,\lambda) = \Big(2\Gamma e_+ - \frac{ic_+\beta_+ u_+}{\sqrt{\tilde\xi^2-\beta_+^2}} + \frac{\Gamma u_+^2}{\beta_+}, \frac{ic_+\beta_+}{\sqrt{\tilde\xi^2-\beta_+^2}} - \Gamma u_+, \frac{c_+\tilde\xi}{\sqrt{\tilde\xi^2-\beta_+^2}}, \Gamma\Big), \tag{5.35}$$

where

$$\beta := \frac{-u\lambda - \sqrt{\lambda^2 + \tilde\xi^2(c^2-u^2)}}{c^2-u^2}. \tag{5.36}$$

Together with our computation of $A_1^- S$ in (5.18), this determines $\ell_+ \cdot A_1^1 S$. Meanwhile, $\Delta := \ell_+ \cdot (\lambda[U] + i\tilde\xi[F_2(U)])$ is computed for the same choice of ℓ_+ in Appendix C, [Z3] (equation displayed below C.36), thus determining $\hat\eta(\tilde\xi,\lambda) = \ell_+ \cdot A_1^1 S/\Delta(\tilde\xi,\lambda)$.

With Proposition 1.19, this gives a straightforward means of numerical determination of multidimensional instability, by plotting the image of $\hat\eta(1, i\tau)$ as τ ranges over the real axis and checking whether or not this curve strikes the nonnegative real axis; however, we shall not carry this out here.

The numerical determination of one- and multi-dimensional stability transitions for ideal and other gas laws would be interesting problems for further investigation. A further very interesting open open problem is to determine analytically the stability transitions as was done for the inviscid shock problem (involving only Δ) in [Er, M]; see Appendix C, [Z3].

A Computation of ℓ_+ in one dimension

In this appendix, we carry out for completeness the computation of ℓ_+ for the one-dimensional Navier–Stokes equations, verifying (5.25). In variables (ρ, u, v, e), the quasilinear hyperbolic

[2] Contributed by K. Jenssen and G. Lyng

part of the equations becomes

$$
\begin{aligned}
&\rho_t + q\cdot\nabla\rho + \rho \mathrm{div} q = 0,\\
\text{(A.1)}\qquad &q_t + q\cdot\nabla q + \rho^{-1}p_\rho\nabla\rho + \rho^{-1}p_e\nabla e = 0,\\
&e_t + q\cdot\nabla e + \rho^{-1}p\,\mathrm{div} u = 0,
\end{aligned}
$$

where $q=(u,v)$ denotes velocity, or, in one dimension,

$$V_t + (uId + M)V_{x_1} = 0,$$

where $V=(\rho,u,v,e)$ and

$$M := \begin{pmatrix} 0 & \rho & 0 & 0\\ \rho^{-1}p_\rho & 0 & 0 & \rho^{-1}p_e\\ 0 & 0 & 0 & 0\\ 0 & \rho^{-1}p & 0 & 0\end{pmatrix},$$

from which we may conclude that $A_1 = S(uId+M)S^{-1}$ for

$$S := \frac{\partial\big(\rho,\rho u,\rho v,\rho(e+u^2/2+v^2/2)\big)}{\partial(\rho,u,v,e)} = \begin{pmatrix} 1 & 0 & 0 & 0\\ u & \rho & 0 & 0\\ v & 0 & \rho & 0\\ e+u^2/2+v^2/2 & \rho u & \rho v & \rho\end{pmatrix},$$

$$S^{-1} := \begin{pmatrix} 1 & 0 & 0 & 0\\ \frac{-u}{\rho} & \frac{1}{\rho} & 0 & 0\\ \frac{-v}{\rho} & 0 & \frac{1}{\rho} & 0\\ \frac{-e+u^2/2+v^2/2}{\rho} & \frac{-u}{\rho} & \frac{-v}{\rho} & \frac{1}{\rho}\end{pmatrix},$$

and thus $\ell_+^* = \tilde\ell_+^* S^{-1}$ for $\tilde\ell_+$ defined as the left eigenvector of M associated with the eigenvalue of smallest real part, $*$ denoting adjoint, or congugate transpose, all quantities to be evaluated at $(\rho,u,v,e)=(1/u_+,u_+,0,e_+)$.

By inspection, $\tilde\ell_+^* = (p_\rho, -\rho c, 0, p_e)$ for $v=0$, where sound speed c is defined as in (5.26), whence

$$\ell_+^* = \tilde\ell_+^* S^{-1} = \Big(p_\rho + cu + \frac{p_e(u^2/2-e)}{\rho}, -\frac{p_e u}{\rho} - c, 0, \frac{p_e}{\rho}\Big)(U_+)$$

as claimed.

References

[AGJ] J. Alexander-R. Gardner-C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves*. J. Reine Angew. Math. 410 (1990) 167–212.

[Ba] G. K. Batchelor, *An introduction to fluid dynamics.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, paperback edition, 1999.

[CHNZ] N. Costanzino, J. Humpherys, T. Nguyen, and K. Zumbrun, *Spectral stability of noncharacteristic boundary layers of isentropic Navier–Stokes equations,* Preprint, 2007.

[Er] J. J. Erpenbeck, *Stability of step shocks,* Phys. Fluids 5 (1962) no. 10, 1181–1187.

[FS] H. Freistühler and P. Szmolyan. *Spectral stability of small-amplitude viscous shock waves in several dimensions*, To appear, Archive for Rat. Mech. Anal.

[GG] Grenier, E. and Guès, O., *Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems*, J. Differential Eqns. 143 (1998), 110-146.

[GMWZ4] Gues, O., Metivier, G., Williams, M., and Zumbrun, K., *Paper 4*, *Navier-Stokes regularization of multidimensional Euler shocks*, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 1, 75–175.

[GMWZ5] Guès, O., Métivier, G., Williams, M., and Zumbrun, K., *Stability of noncharacteristic boundary layers for the compressible Navier-Stokes and MHD equations*, preprint (2008).

[GMWZ6] Gues, O., Metivier, G., Williams, M., and Zumbrun, K., *Viscous boundary value problems for symmetric systems with variable multiplicities*, to appear, J. Diff. Eq.

[GR] Grenier, E. and Rousset, F., *Stability of one dimensional boundary layers by using Green's functions*, Comm. Pure Appl. Math. 54 (2001), 1343-1385.

[GZ] Gardner, R. and Zumbrun, K., *The gap lemma and geometric criteria instability of viscous shock profiles*, CPAM 51. 1998, 797-855.

[Gi] D. Gilbarg, The existence and limit behavior of the one-dimensional shock layer. *Amer. J. Math.* 73:256–274, 1951.

[HLyZ1] Humpherys, J., Lyng, G., and Zumbrun, K., *Spectral stability of ideal-gas shock layers*, Preprint (2007).

[HLyZ2] Humpherys, J., Lyng, G., and Zumbrun, K., *Multidimensional spectral stability of large-amplitude Navier-Stokes shocks*, in preparation.

[HuZ] J. Humpherys and K. Zumbrun, *Spectral stability of small amplitude shock profiles for dissipative symmetric hyperbolic–parabolic systems.* Z. Angew. Math. Phys. 53 (2002) 20–34.

[Kat] T. Kato, *Perturbation theory for linear operators.* Springer–Verlag, Berlin Heidelberg (1985).

[KK] Y. Kagei and S. Kawashima *Stability of planar stationary solutions to the compressible Navier-Stokes equations in the half space,* Comm. Math. Phys. 266 (2006), 401-430.

[KNZ] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space,* Comm. Math. Phys. 240 (2003), no. 3, 483–500.

[M] A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations.* Mem. Amer. Math. Soc. 275 (1983).

[MaZ3] Mascia, C. and Zumbrun, K., *Pointwise Green function bounds for shock profiles of systems with real viscosity,* Arch. Ration. Mech. Anal. 169 (2003), no. 3, 177–263.

[MN] Matsumura, A. and Nishihara, K., *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas,* Comm. Math. Phys., 222 (2001), no. 3, 449–474.

[MZ1] Métivier, G. and Zumbrun, K., *Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems*, Memoirs AMS, 826 (2005).

[MZ2] Métivier, G. and Zumbrun, K., *Hyperbolic Boundary Value Problems for Symmetric Systems with Variable Multiplicities*, J. Diff. Equ., 211 (2005) 61-134.

[N1] T. Nguyen, *Long-time stability of viscous MHD shocks,* preprint (2008).

[N2] T. Nguyen, *Long-time stability of viscous MHD boundary layers,* preprint (2008).

[NZ1] T. Nguyen and K. Zumbrun, *Long-time stability of large-amplitude noncharacteristic boundary layers for hyperbolic parabolic systems*, preprint (2008).

[NZ2] T. Nguyen and K. Zumbrun, *Long-time stability of multi-dimensional noncharacteristic viscous boundary layers*, preprint (2008).

[PZ] Plaza, R. and Zumbrun, K., *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst. 10 (2004) 885–924.

[R3] Rousset, F., *Stability of small amplitude boundary layers for mixed hyperbolic-parabolic systems,* Trans. Amer. Math. Soc. 355 (2003), no. 7, 2991–3008.

[Se1] D. Serre, *La transition vers l'instabilité pour les ondes de chocs multi-dimensionnelles,* Trans. Amer. Math. Soc. 353 (2001) 5071–5093.

[Se2] D. Serre, *Sur la stabilité des couches limites de viscosité,* (French. English, French summary) [Stability of viscosity boundary layers] Ann. Inst. Fourier (Grenoble) 51 (2001), no. 1, 109–130.

[SZ] Serre, D. and Zumbrun, K., *Boundary layer stability in real vanishing-viscosity limit*, Comm. Math. Phys. 221 (2001), no. 2, 267–292.

[Sm] J. Smoller, *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, second edition, 1994.

[YZ] Yarahmadian, S. and Zumbrun, K., *Pointwise Green function bounds and long-time stability of large-amplitude noncharacteristic boundary layers*, Preprint (2008).

[Z1] Zumbrun, K., *Multidimensional stability of planar viscous shock waves*, Advances in the theory of shock waves, 304-516. Progress in Nonlinear PDE, 47, Birkhäuser, Boston, 2001.

[Z3] Zumbrun, K., *Stability of large-amplitude shock waves of compressible Navier–Stokes equations.* Handbook of Fluid Mechanics III, S.Friedlander, D.Serre ed., Elsevier North Holland (2004).

[ZH] Zumbrun, K. and Howard, P., *Pointwise semigroup methods and stability of viscous shock waves*, Indiana Univ. Math. J. 47. 1998, 741-871.

[ZS] Zumbrun, K. and Serre, D. *Viscous and inviscid stability of multidimensional planar shock fronts*, Indiana Univ. Math. J. 48. 1999, 937-992.