Maths pour le climat

FSMP, Paris 5e,
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Les mathématiques au coeur des études climatiques

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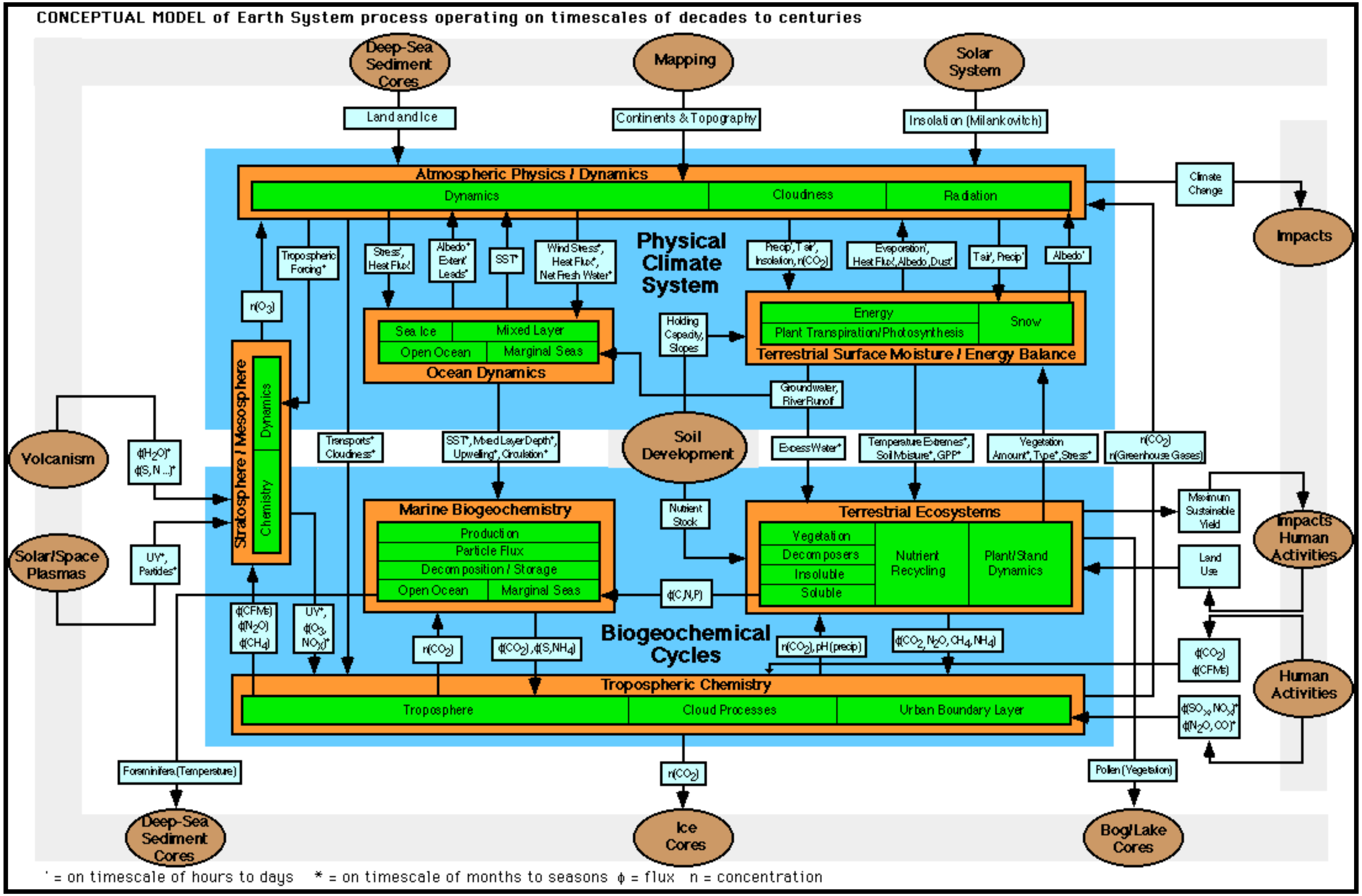
Prière de consulter ces sites pour plus ample informé :
http://www.atmos.ucla.edu/tcd/, http://www.environnement.ens.fr/
et https://www.researchgate.net/profile/Michael_Ghil

F. Bretherton's "horrendogram" of Earth System Science



Earth System Science Overview, *NASA Advisory Council*, 1986

Climate and Its Sensitivity

Let's say CO_2 doubles:

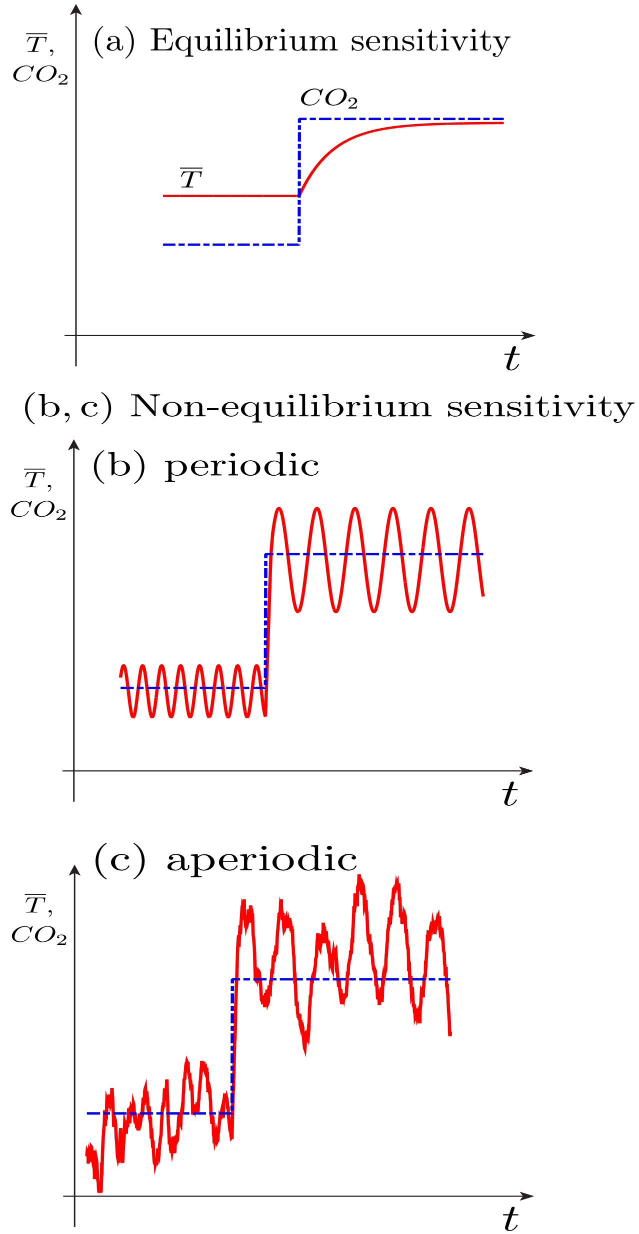
How will "climate" change?

1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)

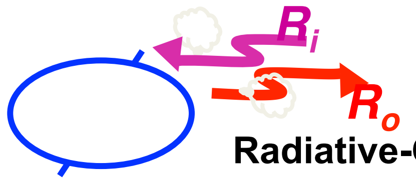


Main points

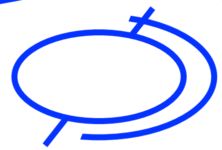
- The ***climate system*** is highly ***nonlinear and*** quite ***complex*** ➔ it is NOT, repeat ***NOT in equilibrium***.
- One cannot separate ***natural variability*** from ***anthropogenic change***, at least not by the naïve, linear methods used so far.
- Thus the ***climate change*** problem is inseparable from the ***climate sensitivity*** and the ***natural variability*** problems.
- We know now, in principle, how to determine ***climate sensitivity*** for a nonlinear, chaotic ***climate system*** subject to both ***natural*** and ***anthropogenic*** forcing.
- This requires both ***deterministic*** & ***stochastic*** methods.
- One needs to develop ***effective algorithms*** for ***computing*** the ***sensitivity*** of large climate models to ***tenths of parameters*** and parametrizations.

Climate models (atmospheric & coupled) : A classification

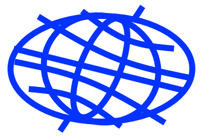
- *Temporal*
  - stationary, (quasi-)equilibrium
  - transient, climate variability

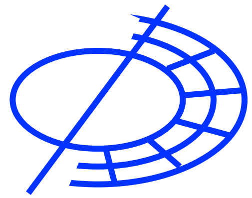


Radiative-Convective Model(*RCM*)



Energy Balance Model (*EBM*)





- *Space*
  - 0-D (dimension 0)
  - 1-D
    - vertical
    - latitudinal
  - 2-D
    - horizontal
    - meridional plane
  - 3-D, GCMs (General Circulation Model)
  - Simple and intermediate 2-D & 3-D models

- *Coupling*
  - Partial
    - unidirectional
    - asynchronous, hybrid
  - Full

➔ ***Hierarchy***: back-and-forth between the simplest and the most elaborate model, and between the models and the observational data

Outline

- ***Tipping Points I*: *Autonomous Systems***
  - – Fixed points, their stability, and transitions
- ***Climate application***
  - – Earth's radiation budget & energy balance models (EBMs)
- ***Tipping Points II*: *Non-autonomous & Random Systems***
  - – Pullback & random attractors
- ***"Toy" application***
  - – The Lorenz (1963) convection model with random forcing
- ***Concluding remarks & bibliography***

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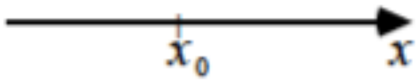
1. Fixed points, I

We start with a scalar ordinary differential equation (ODE)

$$\dot{x} = f(x;\mu)$$

depending on the parameter μ.

Linear stability, $\mu = 1$.



$$f(x_0) = 0 \Rightarrow \dot{x} = 0 \Rightarrow x \equiv x_0$$ – *Fixed point (FP)*

Consider an initial perturbation at $t = 0$:

$$x(0) = x_0 + \xi(0),$$

$$\dot{x} = \dot{x}_0 + \dot{\xi} = \dot{\xi}$$

$$= f(x_0 + \xi) = f(x_0) + f'(x_0)\xi + O(\xi^2)$$

For an infinitesimal perturbation $\xi(0) = \xi_0$

$$\dot{\xi} = f'(x_0)\xi, \qquad f'(x_0) = \lambda, \qquad \dot{\xi} = \lambda\xi,$$

$$\Rightarrow \xi(t) = e^{\lambda t}\xi(0)$$

1. Fixed points, II

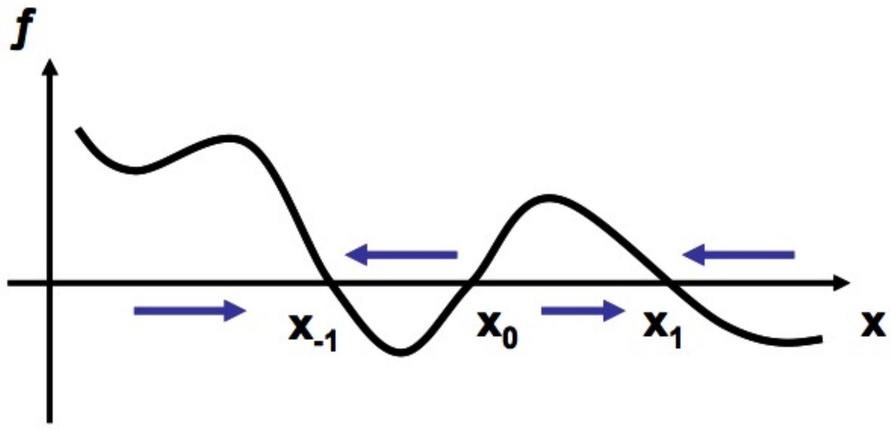
If $\lambda < 0 \Rightarrow$ the fixed point (FP) is (linearly) stable

If $\lambda > 0 \Rightarrow$ the FP is (linearly) unstable

If $\lambda = 0 \Rightarrow$ the linear stability of the FP is neutral

Some basic features on FPs:

1. $f \in C^1, f \neq 0$ on all sub-intervals: FPs are isolated (generic property)
2. Basins of attraction are open intervals (possibly semi-infinite)



2. Saddle-node bifurcations

How does the geometry of the solutions change when $\mu \neq \mu_0$, i.e. how do the number of the stability of the stationary solution change?

Let us start with the scalar case.

A simple case: the saddle-node

$$\dot{x} = \mu - x^2 \equiv f(x;\mu)$$

FPs: $\mu - x^2 = 0 \quad x = \pm\sqrt{\mu}$

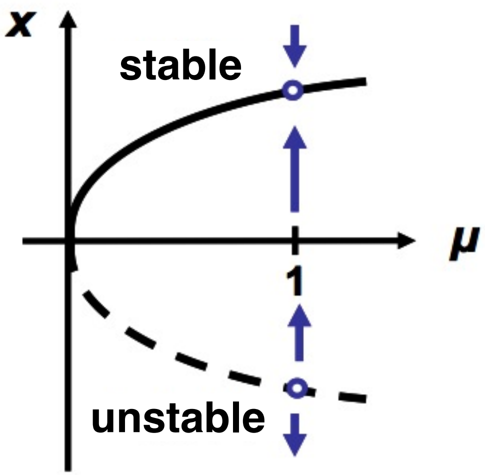
FP stability:

$$x_1 = \sqrt{\mu}, \qquad x_{-1} = -\sqrt{\mu}$$

$$x(0) = x_{\pm 1} + \xi(0)$$

$$\dot{\xi} = \lambda_{\pm}\xi,$$

$$\lambda_{\pm} \equiv f'(x_{\pm 1}) = -2x_{\pm 1} = \mp 2\sqrt{\mu}$$



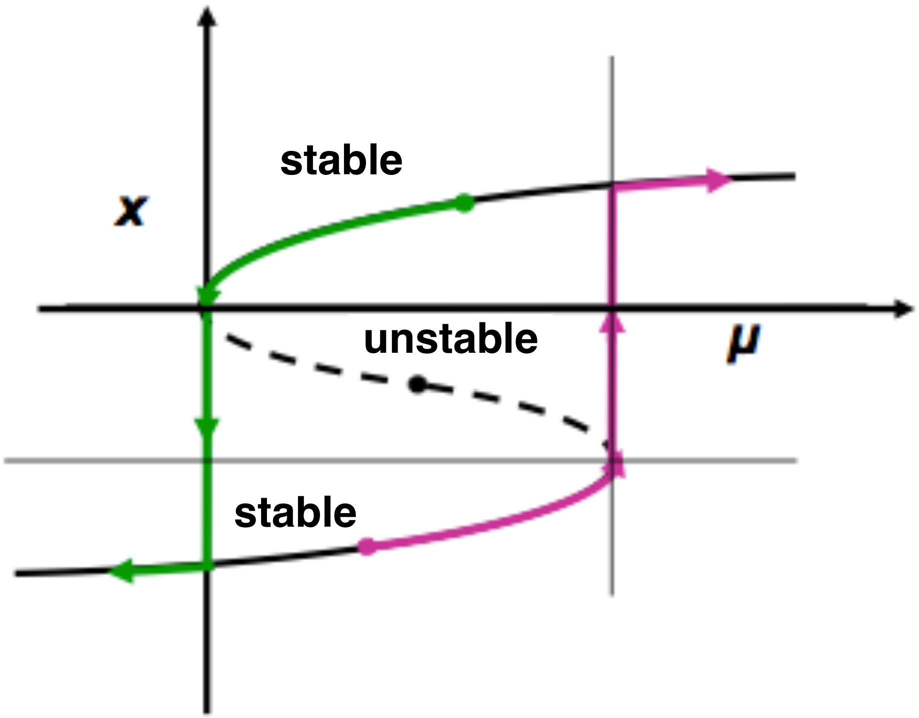
Upper branch (linearly) *stable*

Lower branch (linearly) *unstable*

Let us now examine the nonlinear stability

5. Bistability and hysteresis

The combination of two saddle-node bifurcations can create a hysteresis phenomenon (an *S*-shaped curve) :



$\dot{x} = \mu - x^2$: the top-left bifurcation

$\dot{x} = (\mu - 1) + (x + \frac{1}{2})^2$: the bottom-right bifurcation

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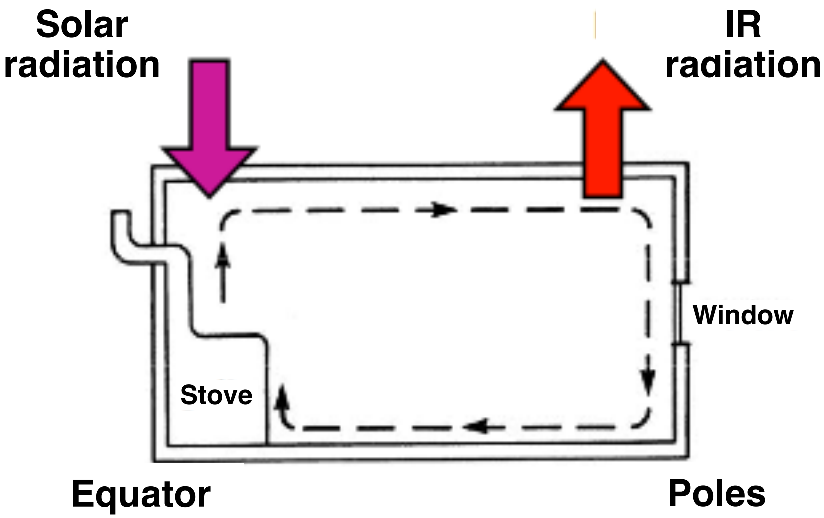
Outline: Climate application

Earth radiation balance + Energy balance models

- Earth radiation budget
  - global + latitude-dependent
- Energy balance models (EBMs)
  - 0-D + 1-D
- Multiple equilibria and their stability
  - snowball Earth
- Transitions and hysteresis
  - Arctic climate and “polar amplification”

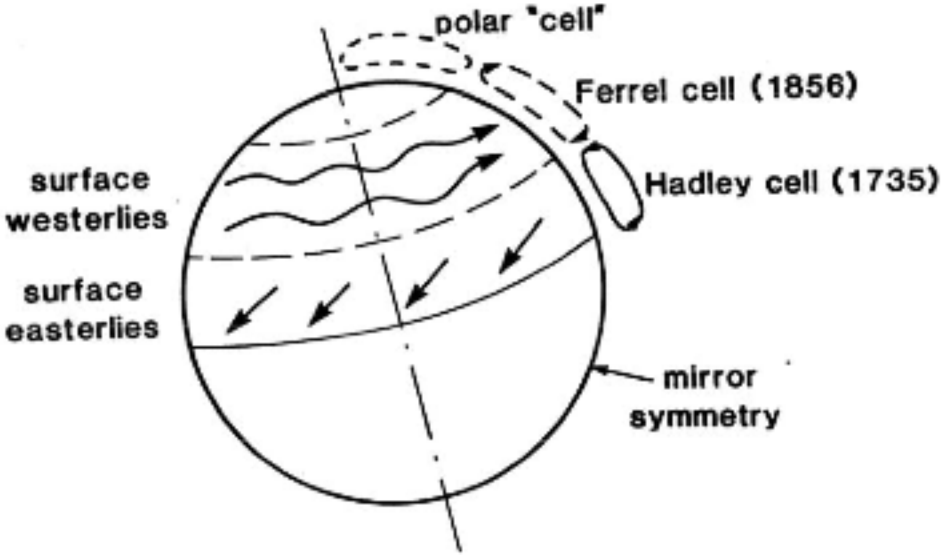
The mean atmospheric circulation

Direct Hadley circulation



Idealized view of the atmosphere's global circulation.*

Observed circulation



Schematic diagram of the atmospheric global circulation.*

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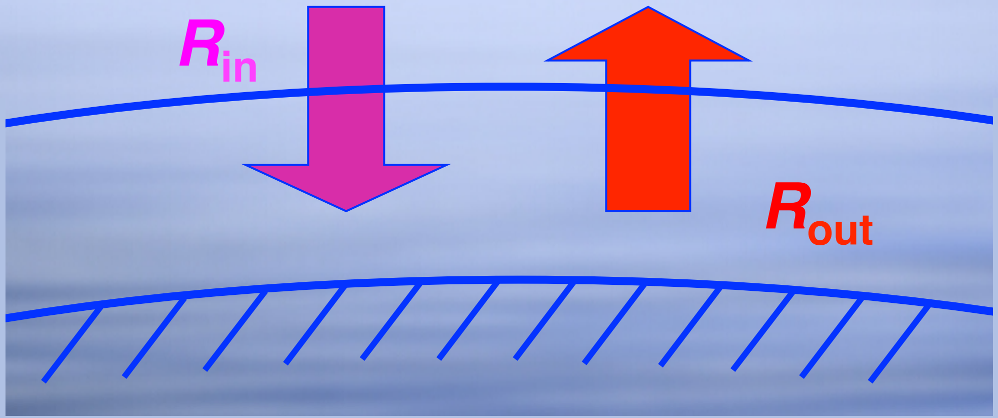
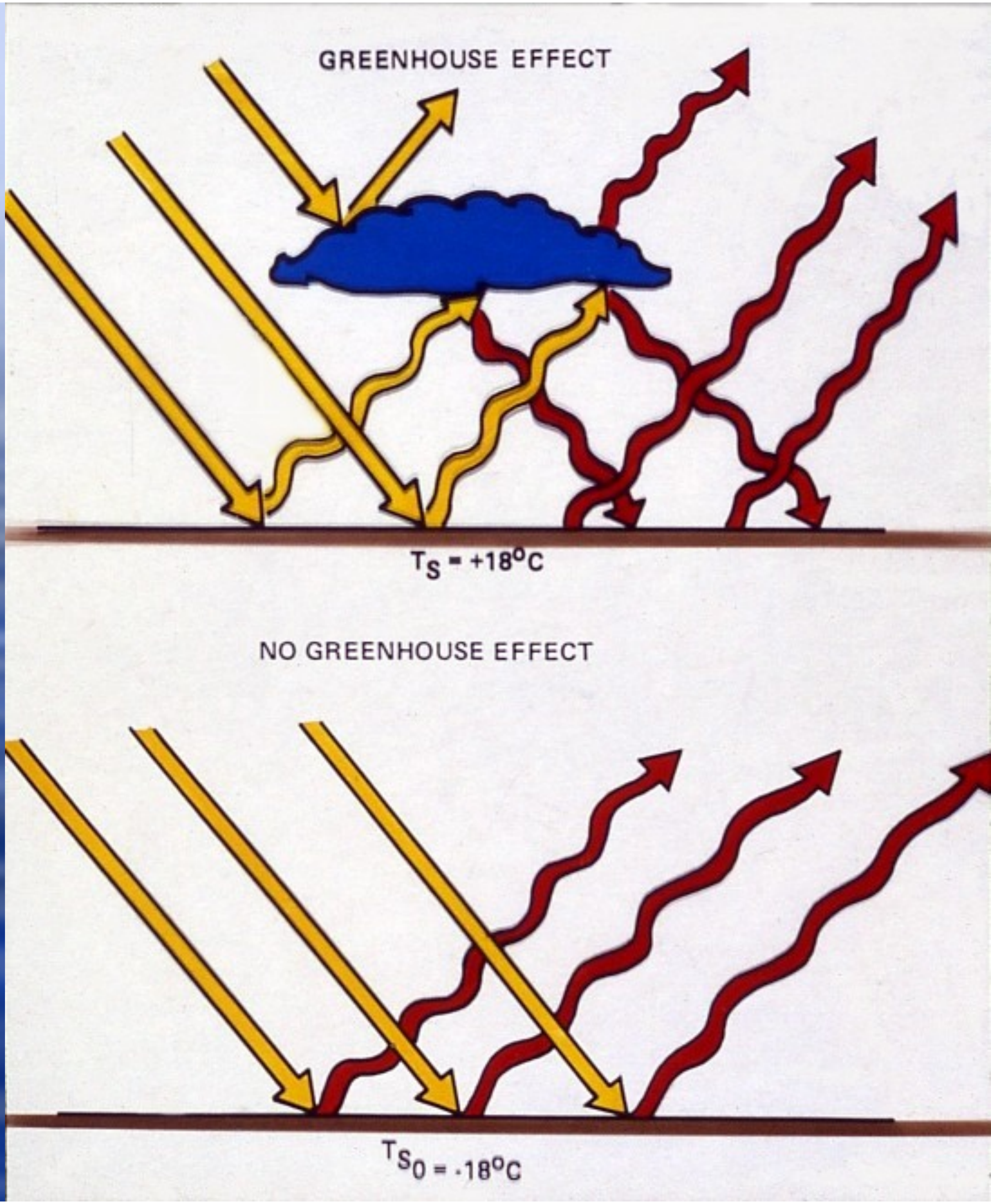
*From **Ghil and Childress (1987), Ch. 4**

Radiative balance



Long-term equilibrium between incident (solar, ultra-violet + visible) radiation R_{in} and outgoing (terrestrial, infrared) radiation R_{out} dominates climate.

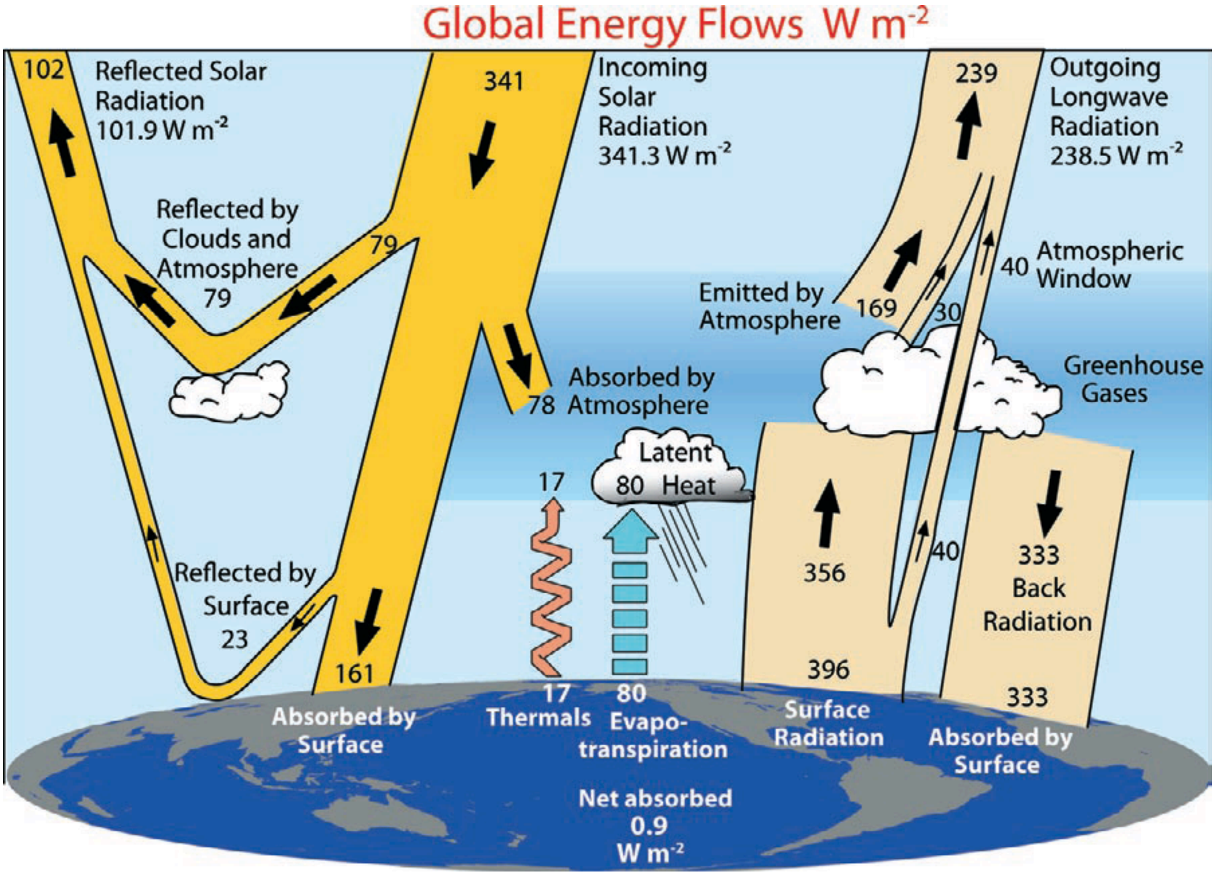
Refs. [1] **Egyptian scribe** (3000 B.C.) :
"The Sun heats the Earth," *Rosetta stone*, ll. 13–17.
[2] **Herodotus** (484 - cca. 425 B.C.)


GREENHOUSE EFFECT
$T_S = +18^oC$
NO GREENHOUSE EFFECT
$T_{S_0} = -18^oC$

Earth's Global Energy Budget

K.E. Trenberth, J.T. Fasullo & J. Kiehl, 2009,

Bull. Amer. Meteorol. Soc., **90**(3), 311–323.



Energy-balance models (EBMs)

$$C\frac{\partial T}{\partial t} = R_i - R_o + D$$

C — local calorific capacity

T — local surface temperature

R_i — incident solar radiation

R_o — terrestrial radiation towards space

D — heat redistribution ('diffusion')

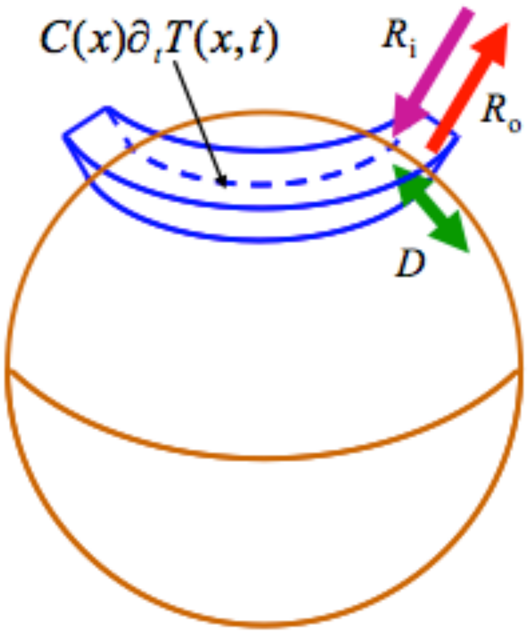
Comments:

1. C, R_i, R_o and D have to be calculated ("parameterized") according to $T = T(x,t)$

2. The model's main characteristic is R_i

$$R_i = Q(x)\{1 - \alpha(x,T)\}$$

with α the local albedo.



0-D version (averaged over the globe)

$$C\frac{d\bar{T}}{dt} = R_i - R_o = Q\left\{1-\alpha(\bar{T})\right\} - \sigma\bar{T}^4 m(\bar{T})$$

$\bar{T}$ — average surface temperature

t — time (in thousands of years)

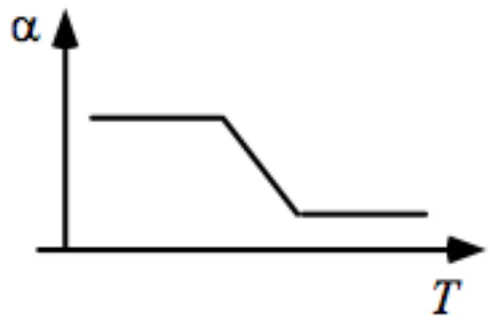
Q — incident solar flux

α — albedo

C — calorific capacity

σ — Stefan–Boltzmann constant

m — greenhouse effect factor



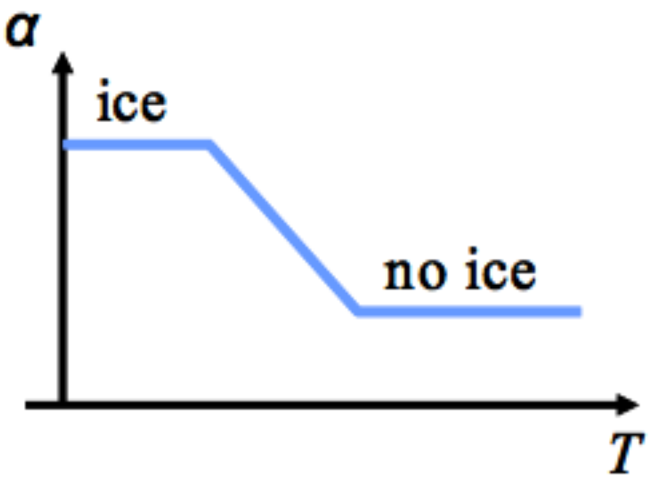
Comments:

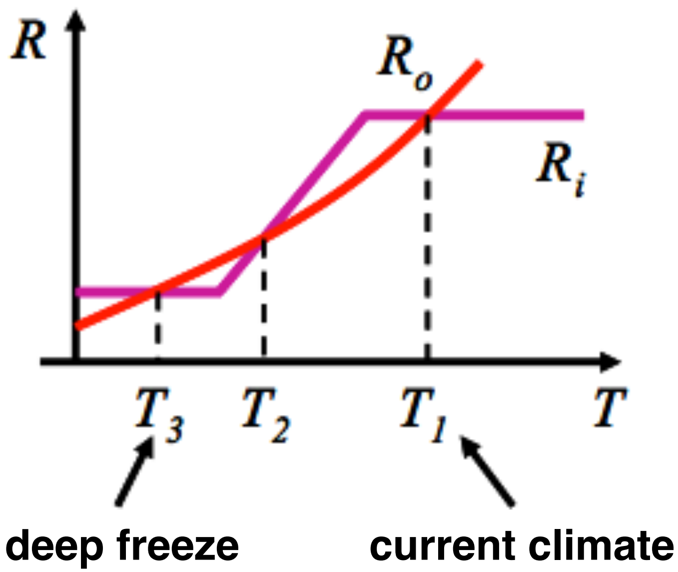
α depends on the ice and snow cover, on cloud cover, etc. (implicit variables). All is parameterized as a function of the explicit variable $\bar{T}$.

0-D EBM, I: Model solutions

We want to write T as: $T = T(t; T_0, Q, c, ...)$

Stationary solutions: $Q\{1-\alpha(T)\} - \sigma T^4 = 0$





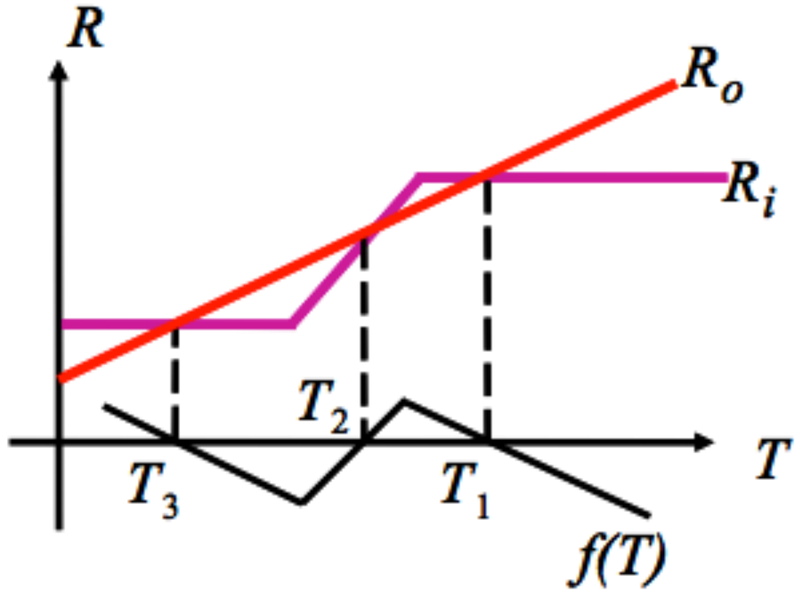
What happens if the sun "blinks" and $T = T_1 + \Delta T$?

We have to go back to the original equation, which depends on time.

0-D EBM, II: Stability condition

$$C\partial_t T = R_i - R_o = f(T)$$

$$R_i = Q\{1 - \alpha(T)\}$$
$$R_o = A + BT$$



We set $T = T_j + \theta$:

$$f(T_j) = 0,$$
$$f(T) = f(T_j) + f'(T_j)\theta + \ldots$$

Let's define $\lambda_j \equiv f'(T_j)/c$

$$\partial_t\theta = \lambda_j\theta \Rightarrow \theta = e^{\lambda_j t}\theta_0$$

If $\lambda_j < 0$ stable;

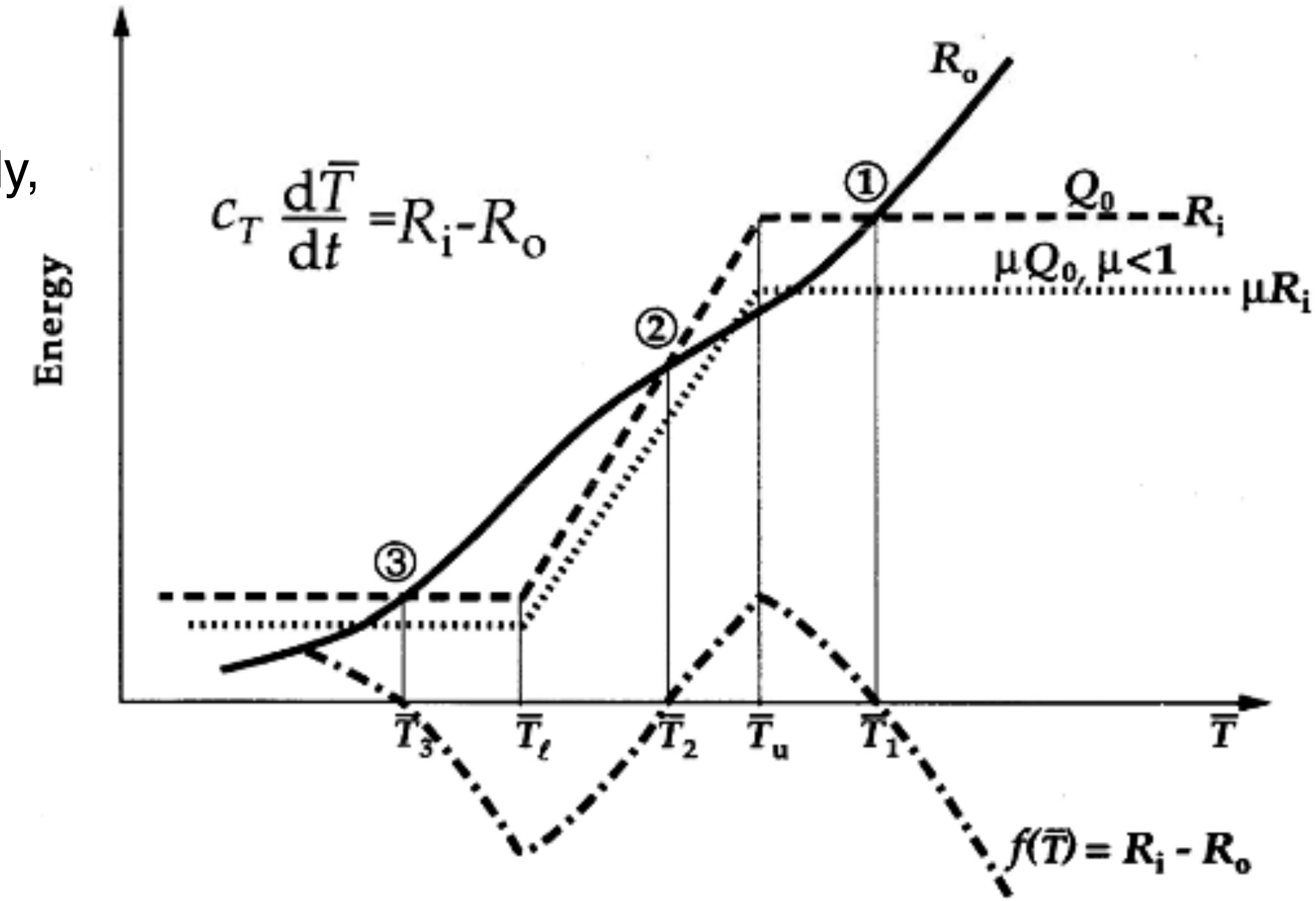
if $\lambda_j > 0$ unstable.

Comment: in the 1-D case $\lambda_j \to \lambda_j^{(0)}$;
$\lambda_j \sim 1/c$

0-D EBM, III: Changes in parameters

What happens if the insolation parameter μ changes, i.e., the "solar constant" changes? This may represent a change in solar luminosity, orbital parameters or in the optical properties of the atmosphere.

❖The model's three "climates" shift in value and, possibly, in number.



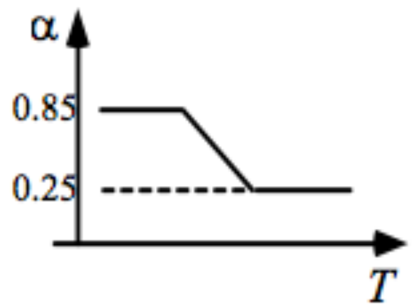
1-D version ('classic' EBM)

$$C(x)T_t = R_i - R_o + D$$

T — temperature
x — latitudinal coordinate
$\tilde{T}(x)$ — the observed climate

Boundary conditions: $T_x(0) = T_x(1) = 0$
$x = 0$ Pole (North)
$x = 1$ Equator

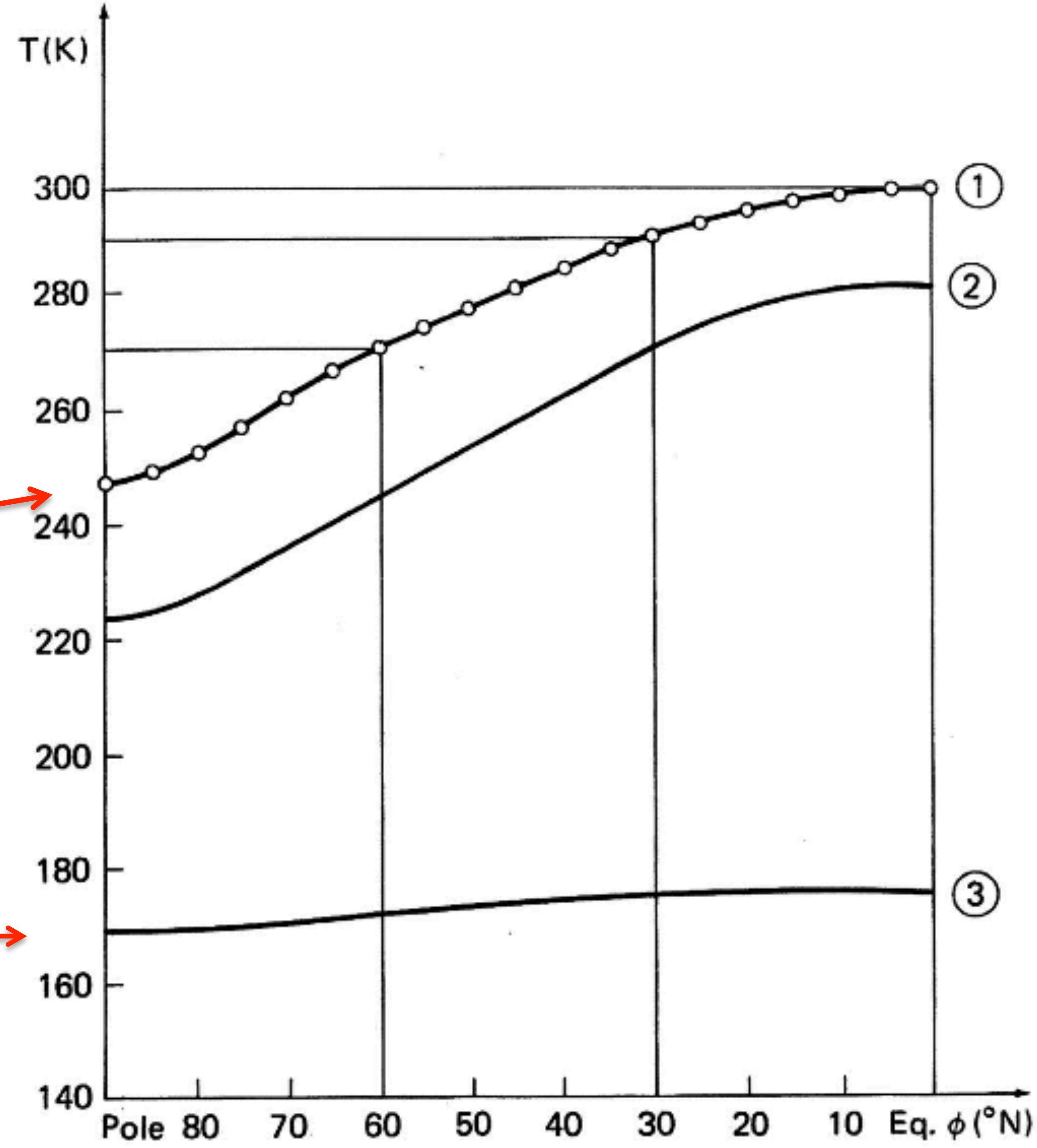
$$R_i = Q(x)\{1 - \alpha\}$$
$$= Q(x)\{1 - b(x) + c_1 T\}_c$$
$$R_o = \sigma T^4\{1 - m\tanh(c_3 T^6)\}$$



$$D = \frac{1}{\sin\frac{\pi x}{2}}\partial_x \sin\frac{\pi x}{2}\{k(x) + k_s(x)g(\tilde{T})\}T_x$$

Questions: 1. Stationary solutions ('climates')?
2. Stability?
3. Perturbation & bifurcation? $Q \to \mu Q \quad (\mu = 1)$

The three climates of the 1-D model



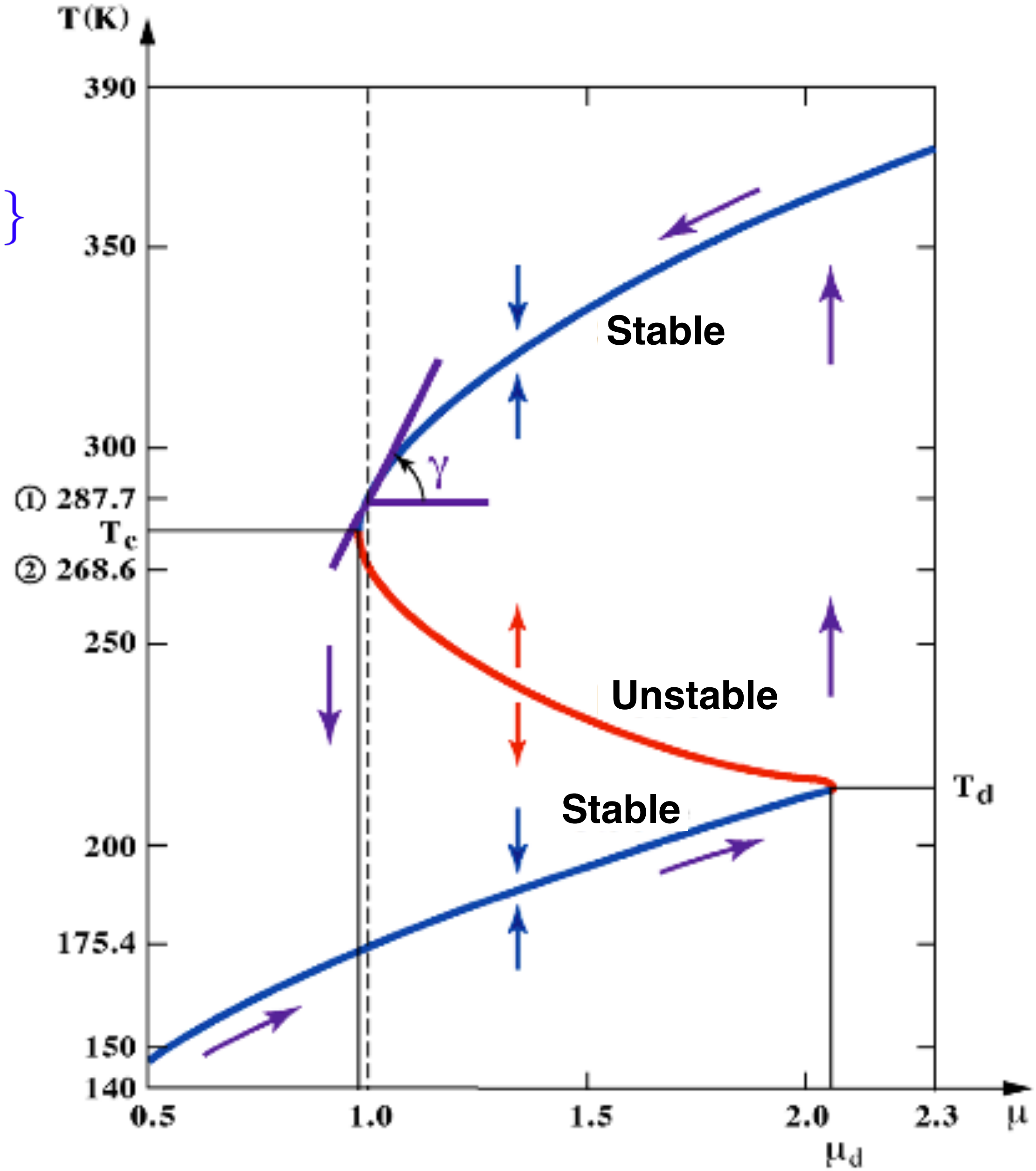
Present climate

"Deep freeze" or snow-ball Earth

1-D EBM: Bifurcation diagram

$$C(x)T_t = \{k(x,T)T_x\}_x$$
$$+\mu Q_0\{1-\alpha(x,T)\}$$
$$-g(T)\sigma T^4$$

$T_x = 0$ at $x = 0, 1$



Climate sensitivity:

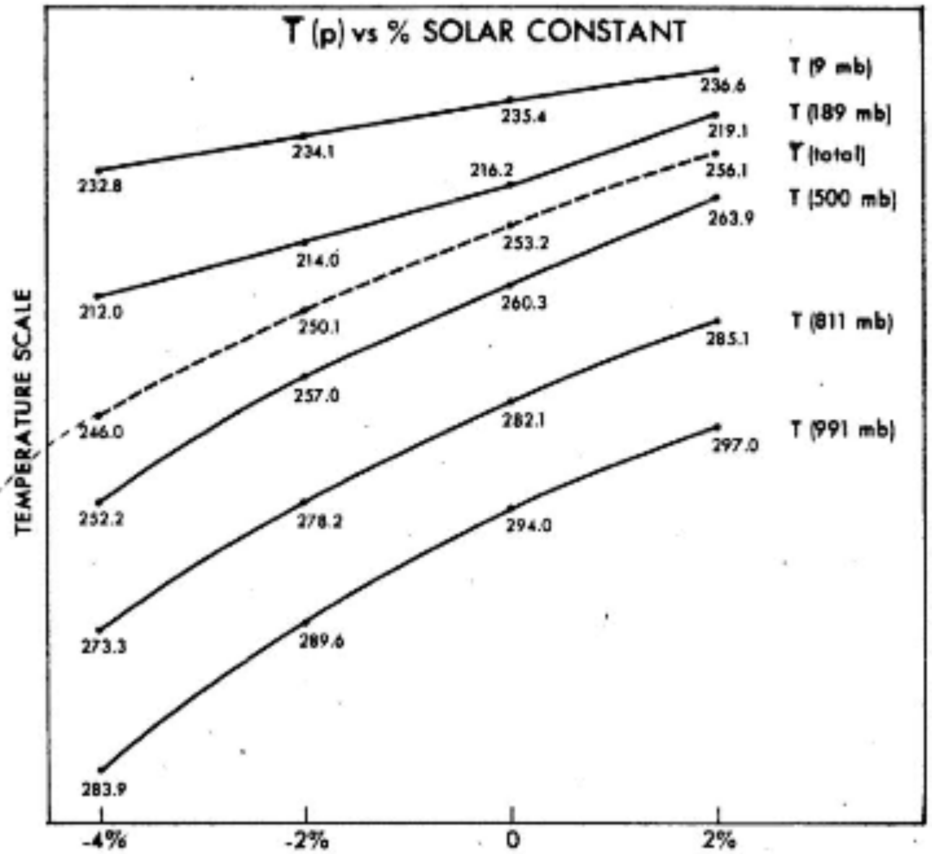
$$\gamma = \frac{dT}{d\mu} \cong 0.01$$

(1K per % of Q)

Climate sensitivity to insolation in a General Circulation Model (GCM)

"As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization."*

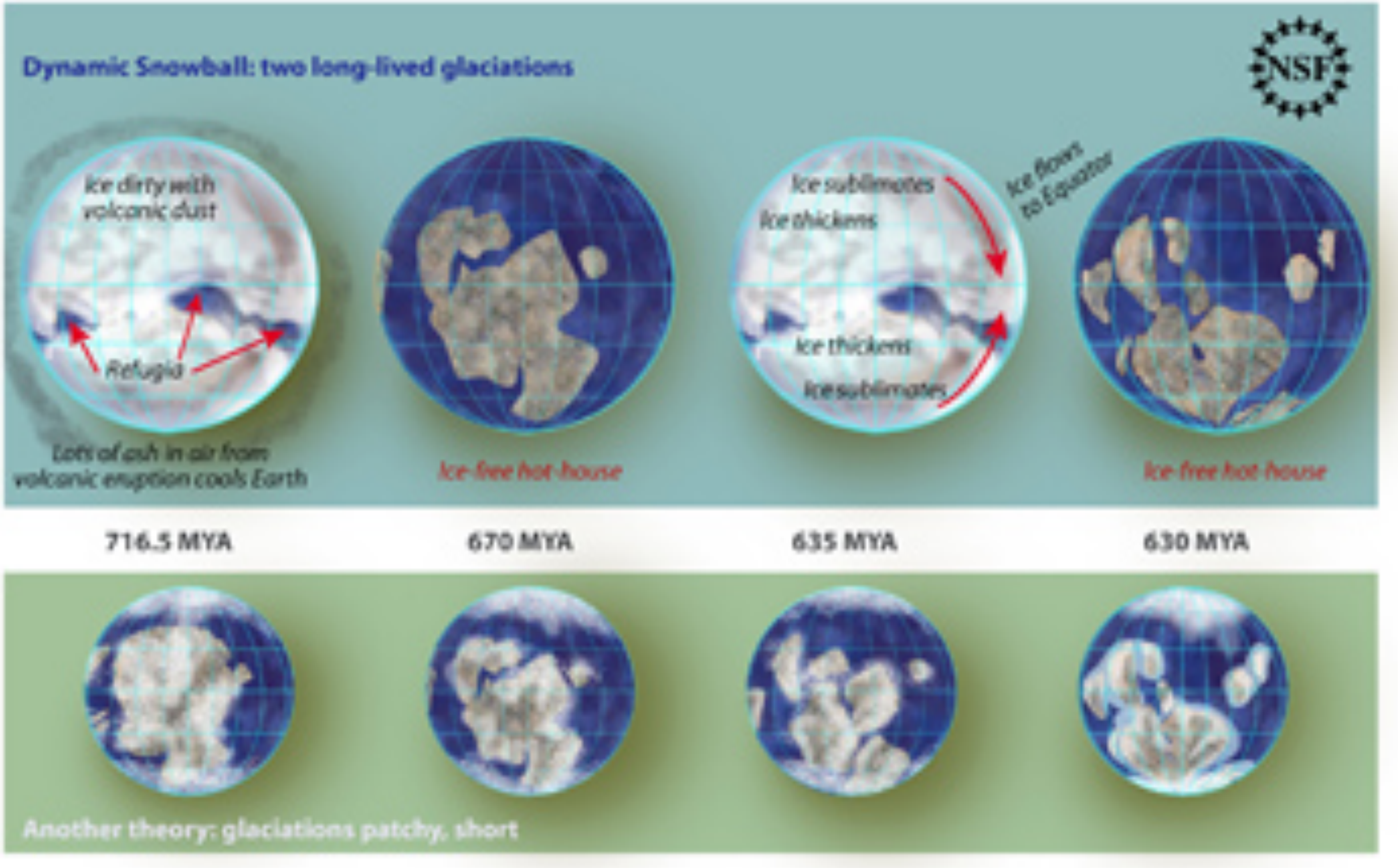
"Nevertheless, it seems to be significant that both the one-dimensional and three- dimensional models yield qualitatively similar results in many respects."*



Area-mean temperatures for various model levels, as well as a mass-weighted mean temperature for the total model atmosphere. Based on 4 GCM runs: control, –4%, –2% and +4%. Units are in degrees K.

* From Wetherald and Manabe, 1975, *J. Atmos. Sci.*, **32,** 2044–2059.

Snowball Earth — Erstwhile a “theory”; now a “fact”?



https://www.nsf.gov/news/news_images.jsp?cntn_id=116410&org=NSF

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Time-dependent forcing

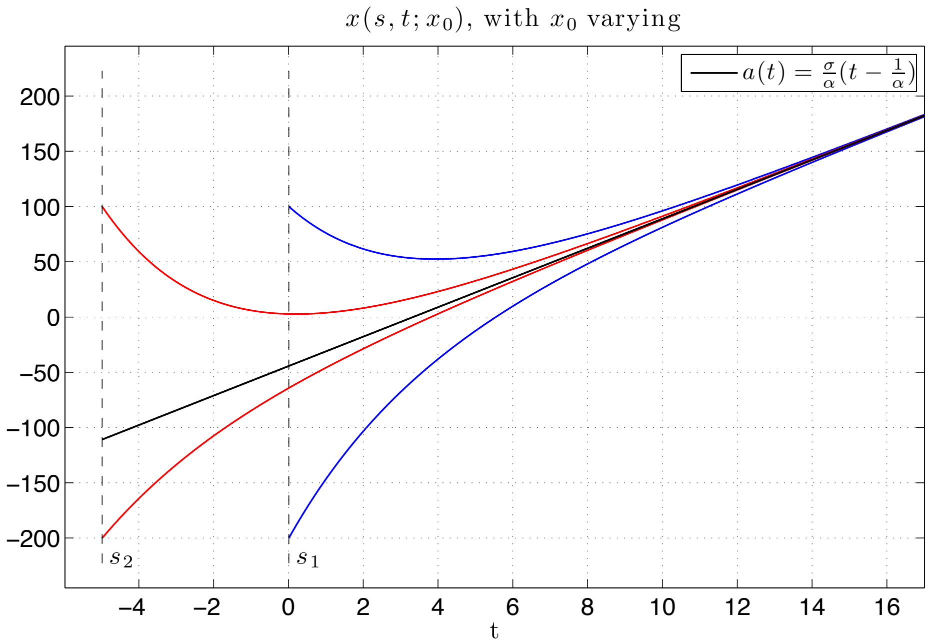
- Much of the theoretical work on the intrinsic variability of the climate system has been done with time-independent forcing and coefficients.
- Mathematically, this relied on autonomous dynamical systems (DDS).
- To address the changes in time of the system's overall behavior — and not just of its mean properties — an important step is to examine time-dependent forcing and coefficients.
- The proper framework for doing so is the theory of non-autonomous and random dynamical systems (NDS and RDS).
- Here is a "super-toy" introduction to **pullback attractors**: what are they?

The pullback attractor of a linear, scalar ODE,

$$\dot{x} = -\alpha x + \sigma t, \quad \alpha > 0, \ \sigma > 0\,,$$

is given by

$$a(t) = \frac{\sigma}{\alpha}\left(t - \frac{1}{\alpha}\right).$$



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The Lorenz model (1963a): a concrete example of a strange attractor(*)

- ***The model equations: 3 coupled, nonlinear ODEs***

$$\dot{X} = \quad -\sigma X \quad +\sigma Y \tag{1}$$

$$\dot{Y} = \quad -XZ \quad +rX \quad -Y \tag{2}$$

$$\dot{Z} = \quad XY \quad -bZ \tag{3}$$

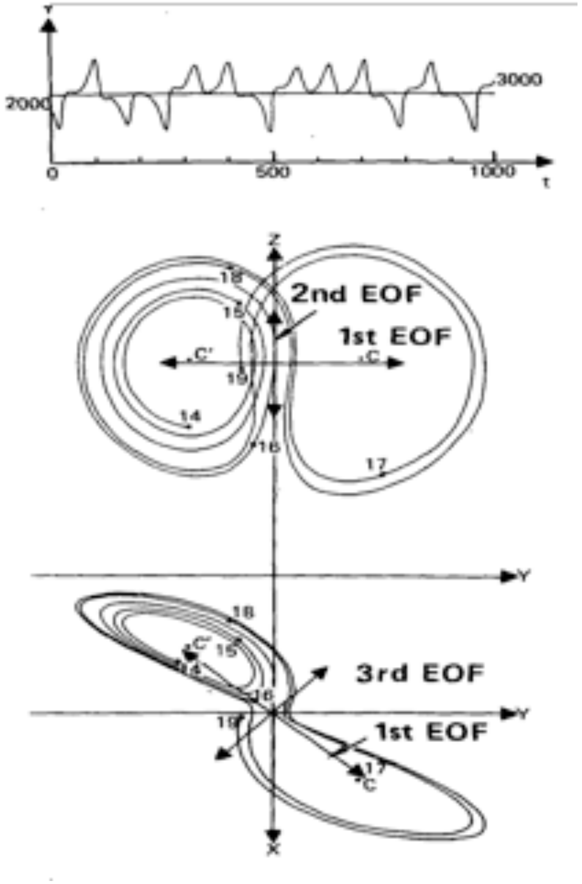
- ***Physics: a model of thermal convection in 2-D***

The variables *X* and *Y* represent the intensity of the velocity field in a 2-D space, *Z* is the deviation of the vertical temperature profile from pure conduction (no motion), and (*X, Y, Z*)$^\bullet$ is their rate of change.

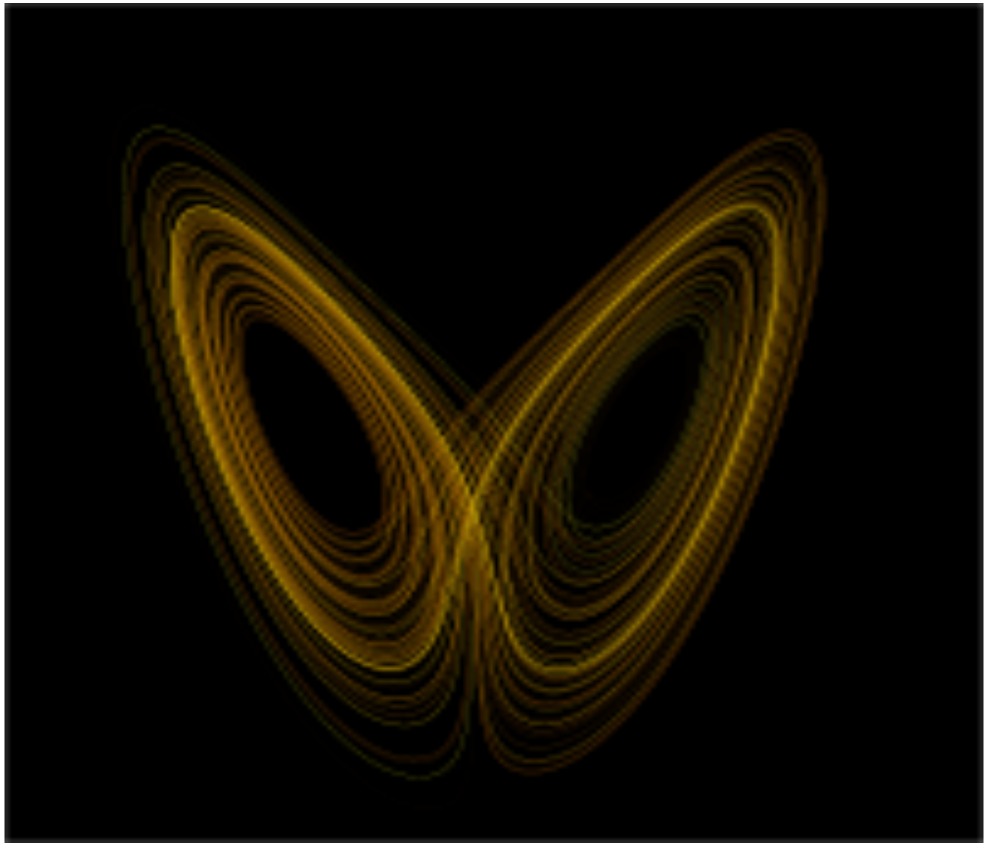
The parameters are the Rayleigh number *ρ* (intensity of the thermal forcing), the Prandtl number σ (the fluid's dissipative properties) and *β* caracterizes the wave length of the perturbation from pure conduction.

(*) Mommy, what's a strange attractor, please?

The Lorenz model (1963a) – some numerical solutions



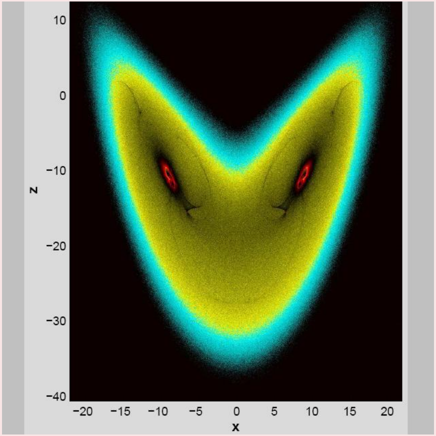
Plot of $Y = Y(t)$ + projections onto the (X, Y) & (Y, Z) planes



Trajectory in phase space

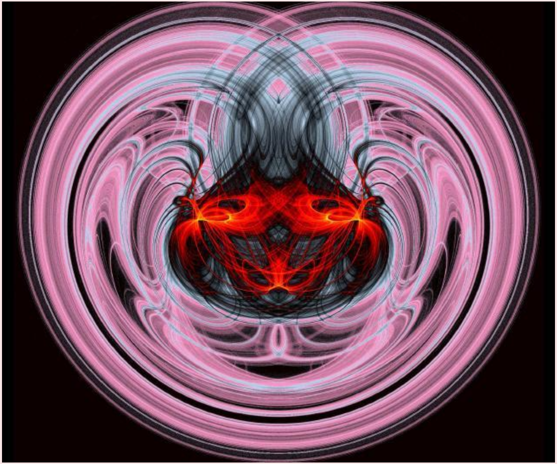
Both for the canonical “chaotic” values $\rho = 28$, $\sigma = 10$, $\beta = 8/3$.

Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time t, and for a fixed realization ω. We show a “projection”, $\int \mu_\omega(x, y, z)\mathrm{d}y$, with multiplicative noise: $\mathrm{d}x_i$=Lorenz$(x_1, x_2, x_3)\mathrm{d}t + \alpha\ x_i \mathrm{d}W_t; i \in \{1, 2, 3\}$.
- 10 million of initial points have been used for this picture!

Sample measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

A day in the life of the Lorenz (1963) model's random attractor; see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*) or https://vimeo.com/240039610

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Conclusions

- Bifurcations ➔ équilibres multiples.
- Points de basculements ➔ transitions rapides entre ces points.
- Applications climatiques :
  - la Terre boule de neige
  - un ocean arctique sans glace ?!
- Transitions entre des types de comportement plus complexes (cycles–limite, attracteurs étranges) – la prochaine fois.
- Aujourd'hui – ***B-tipping***; la prochaine fois également ***N-tipping*** et ***R-tipping***.
- Prévision des transitions ➔ sujet de débats intenses !

Some general references:
Closed + open dynamical systems and climate

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