Global Minimum for a Finsler Elastica Approach

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Motivation and Contributions

- Active Contours Model has been widely used for medical imaging and computer vision which aims to minimize the curve energy:

$$E_{\text{Snake}}(\Gamma)=\int_0^1\Big(\underbrace{w_1\|\dot\Gamma(t)\|^2+w_2\|\ddot\Gamma(t)\|^2}_{\text{Regularization}}+\underbrace{P(\Gamma(t))}_{\text{Image Data}}\Big)dt\Rightarrow\begin{cases}\bullet\ \text{Sensitive to initialization}\\ \bullet\ \text{Sensitive to local minimum}\end{cases}$$

- Isotropic Riemannian metric based minimal path model finds the global minimum of

$$\ell(\gamma)=\int_0^1\Big(\omega+P(\Gamma(t))\Big)\|\dot\Gamma(t)\|dt\xrightarrow{\text{Advantages}}\begin{cases}\bullet\ \text{Global minimum of }\ell\\ \bullet\ \text{Efficient fast marching method}\end{cases}$$

- From E_{Snake} to ℓ: Remove $\|\ddot\Gamma\|$ associated to curvature.

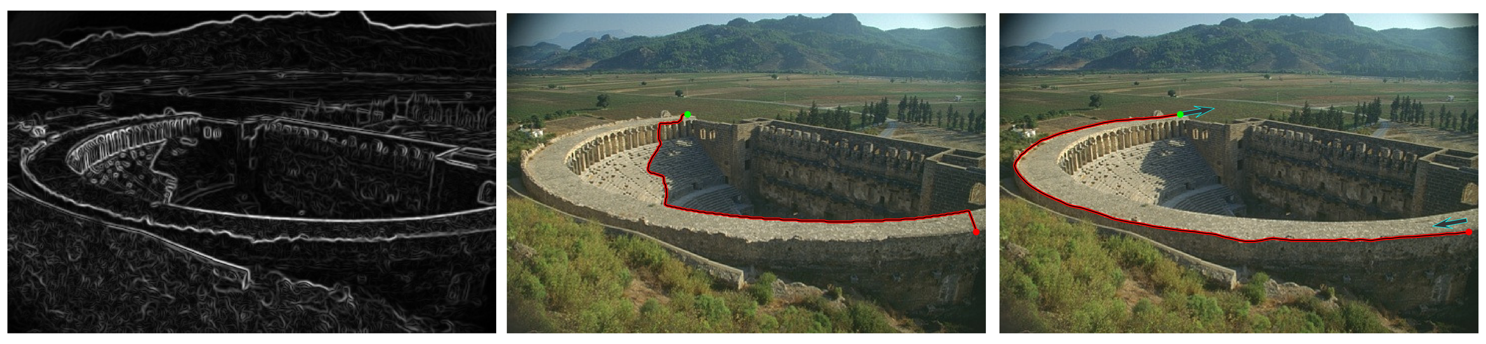

Fig. 1: **Left**: Edge saliency map. **Middle**: Minimal path without curvature. **right**: Elastica minimal path.

Contributions: Introduce the curvature penalty to Eikonal PDE-based minimal path framework by interpreting Euler elastica curve by a geodesic associated to a Finsler elastica metric so that the extracted minimal path is smooth.

Finsler metrics with Randers forms

1. A Finsler metric $\mathcal{F}$ with Rander form can be defined by a vector field $\vec\omega$ and a positive symmetric definite tensor field $\mathcal{M}$:
   - $\mathcal{F}(\mathbf{x},\vec u)=\sqrt{\langle\vec u,\mathcal{M}(\mathbf{x})\vec u\rangle}-\langle\vec\omega(\mathbf{x}),\vec u\rangle,\ s.t.\ \ \langle\vec\omega(\mathbf{x}),\mathcal{M}^{-1}(\mathbf{x})\vec\omega(\mathbf{x})\rangle<1,\ \forall\mathbf{x}\in\Omega\subset\mathbb{R}^d.$
   - $\mathcal{F}$ is an anisotropic Riemannian metric if $\vec\omega=\mathbf{0}$:
   $$\mathcal{F}(\mathbf{x},\vec u)=\sqrt{\langle\vec u,\mathcal{M}(\mathbf{x})\vec u\rangle}$$
   - $\mathcal{F}$ is an isotropic Riemannian metric if $\mathcal{M}$ is proportional to a diagonal matrix.
2. Curve length associated to $\mathcal{F}$ for a regular curve $\gamma:[0,1]\to\Omega$
$$\mathcal{L}(\gamma)=\int_0^1\mathcal{F}(\gamma(t),\dot\gamma(t))\,dt$$
3. Minimal action map $\mathcal{U}(\mathbf{x}):=\min\{\mathcal{L}(\gamma);\,\gamma(0)=\mathbf{s},\gamma(1)=\mathbf{x}\}$
4. Eikonal PDE associated to the metric $\mathcal{F}$:
$$\begin{cases}\mathcal{F}^*(\mathbf{x},\nabla\mathcal{U}(\mathbf{x}))=1, & \forall\mathbf{x}\in\Omega\\ \mathcal{U}(\mathbf{s})=0.\end{cases}\qquad\text{where}\quad\mathcal{F}^*(\mathbf{x},\vec u):=\sup_{\vec v\neq\mathbf{0}}\frac{\langle\vec u,\vec v\rangle}{\mathcal{F}(\mathbf{x},\vec u)},\quad\forall\vec u\in\mathbb{R}^d$$

Euler Elastica Bending Energy

1. Let $\Gamma:[0,1]\to\Omega$ and $\theta:[0,1]\to[0,2\pi)$.
$$\frac{\dot\Gamma(t)}{\|\dot\Gamma(t)\|}:=(\cos\theta(t),\sin\theta(t))\quad\Rightarrow\quad\kappa(t)=\frac{\dot\theta(t)}{\|\dot\Gamma(t)\|},\quad\forall\,t\in[0,1]$$
2. κ is the curvature of Γ and $ds=\|\Gamma'(t)\|dt$. An elastica is a path minimizing
$$\ell(\Gamma):=\int_0^L(1+\alpha\,\kappa^2)ds=\int_0^1(1+\alpha\kappa^2)\|\dot\Gamma(t)\|dt=\int_0^1\Big(\|\dot\Gamma(t)\|+\alpha\frac{|\dot\theta(t)|^2}{\|\dot\Gamma(t)\|}\Big)dt$$

Interpreting Elastica Bending Energy $\mathcal{L}$ By A Metric

Notations:
- Orientation lifted domain $\bar\Omega:=\Omega\times[0,2\pi)$. $\bar{\mathbf{x}}=(\mathbf{x},\theta)\in\Omega$ and $\bar{\mathbf{u}}=(\vec u,\nu)\in\mathbb{R}^2\times\mathbb{R}$.
- Orientation lifted curve $\gamma=(\Gamma,\theta):[0,1]\to\bar\Omega$.

1. A metric $\mathcal{F}^\infty:\bar\Omega\times\mathbb{R}^3\to\mathbb{R}^+$ is defined as
$$\mathcal{F}^\infty(\bar{\mathbf{x}},\bar{\mathbf{u}}):=\begin{cases}\|\vec u\|+\alpha\frac{|\theta'|^2}{\|\vec u\|}, & \vec u\propto(\cos\theta,\sin\theta)\\ +\infty, & \text{otherwise}\end{cases}\xrightarrow{\gamma:=(\Gamma,\theta)}\mathcal{L}(\Gamma)=\int_0^1\mathcal{F}^\infty(\gamma(t),\gamma'(t))dt.$$
2. The Finsler metric $\mathcal{F}^\infty$ is too singular for geodesic distance computation. We proposed the following Finsler elastica metric with respect to a parameter $\lambda\gg1$
$$\mathcal{F}^\lambda(\mathbf{x},\vec u):=\sqrt{\lambda^2\|\vec u\|^2+2\alpha\lambda|\nu|^2}-(\lambda-1)\langle(\cos\theta,\sin\theta),\vec u\rangle$$
3. Convergence
$$\begin{aligned}\mathcal{F}^\lambda(\mathbf{x},\vec u)&=\sqrt{\lambda^2\|\vec u\|^2+2\alpha\lambda|\nu|^2}-(\lambda-1)\langle(\cos\theta,\sin\theta),\vec u\rangle\\&=\lambda\|\vec u\|\sqrt{1+\alpha\frac{2|\nu|^2}{\lambda\|\vec u\|^2}}-(\lambda-1)\langle(\cos\theta,\sin\theta),\ \vec u\rangle\\&=\lambda\|\vec u\|(1+\alpha\frac{|\nu|^2}{\lambda\|\vec u\|^2}+\mathcal{O}(\lambda^{-2}))-(\lambda-1)\langle(\cos\theta,\sin\theta),\ \vec u\rangle\\&=\|\vec\mu\|+\frac{\alpha|\nu|^2}{\|\vec\mu\|}+(\lambda-1)(\|\vec u\|-\langle(\cos\theta,\sin\theta),\vec u\rangle)+\mathcal{O}(\lambda^{-1})\end{aligned}$$

$$\mathcal{F}^\lambda(\mathbf{x},\vec u)\xrightarrow{\lambda\to\infty}\mathcal{F}^\infty(\mathbf{x},\vec u)$$

Visualize the Metrics by their unit balls

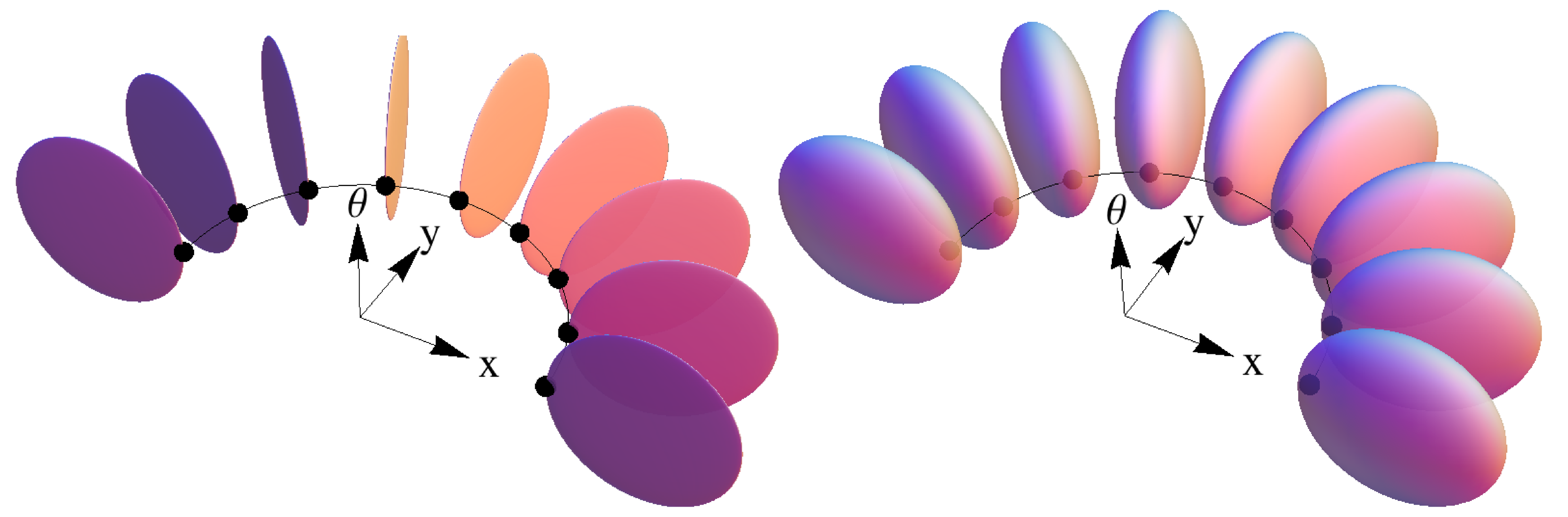

Fig.2 Unit Balls for $\mathcal{F}^\infty$ (left) and $\mathcal{F}^\lambda$ (right) with different orientations. Unit Ball: $B_{\mathbf{x}}=\{\vec u;F(\mathbf{x},\vec u)\le1\}$.

Data-Driven Finsler Metric $\mathcal{P}$ and Fast Marching Method

1. Data-driven Finsler elastica metric:
$$\mathcal{P}(\bar{\mathbf{x}},\cdot)=\frac{1}{\Phi(\bar{\mathbf{x}})}\mathcal{F}^\lambda(\bar{\mathbf{x}},\cdot),$$
$\Phi:\bar\Omega\to\mathbb{R}^+$ is a data-driven function.
2. We make use of the adaptive stencil based fast marching method for the geodesic distance computation with respect to the 3D Finsler elastica metric $\mathcal{P}$, for which we constructed stencils by adapting the construction method for 2D Finsler metric. This method is based on the following Bellman's optimality principle which states that:
$$\mathcal{U}(\mathbf{x})=\min_{\mathbf{y}\in\partial S(\mathbf{x})}\{d_{\mathcal{P}}(\mathbf{x},\mathbf{y})+\mathcal{U}(\mathbf{y})\}$$
where $d_{\mathcal{P}}(\mathbf{x},\mathbf{y})$ denotes the geodesic distance between $\mathbf{x}$ and $\mathbf{y}$. Bellman's optimality principle can be approximated by the Hopf-Lax update:
$$\mathcal{U}(\mathbf{x})=\min\{\mathcal{P}(\mathbf{x},\mathbf{y}-\mathbf{x})+\mathrm{I}_{S(\mathbf{x})}\mathcal{U}(\mathbf{y})\},$$
where $\mathrm{I}_{S(\mathbf{x})}$ denotes the piecewise interpolation operator on the neighbourhood $S(\mathbf{x})$ of $\mathbf{x}$.

Computation time (in seconds) and average number of Hopf-Lax updates required for each grid point by fast marching method with $\alpha=500$ and different values of λ on a $300^2\times108$ grid.

λ	1	10	20	30	100	200	1000
time	13.9s	25.3s	27.3s	27.7s	31.7s	33.9s	36.8s
number	3	5.49	6.06	6.49	7.27	7.82	8.12

Smoothness and Asymmetry

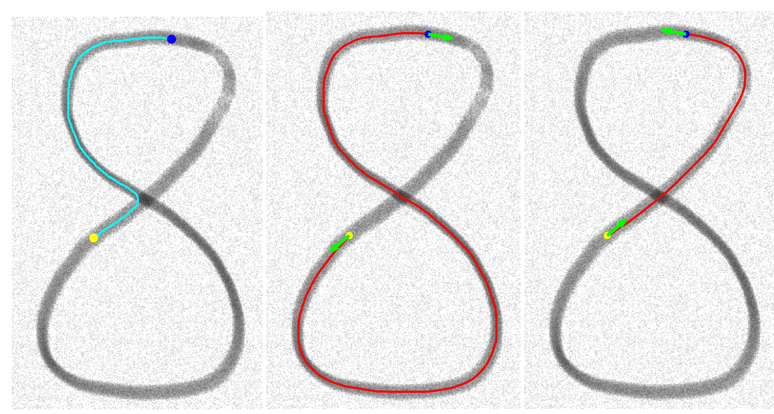

Fig. 3. **Left**: Path using isotropic Riemannian metric. **Middle** and **Right**: Paths using Finsler elastica metric.

Fig. 4. Minimal paths for boundaries extraction using the Finsler elastica metric $\mathcal{P}$.

Contour Detection for Segmentation and Grouping

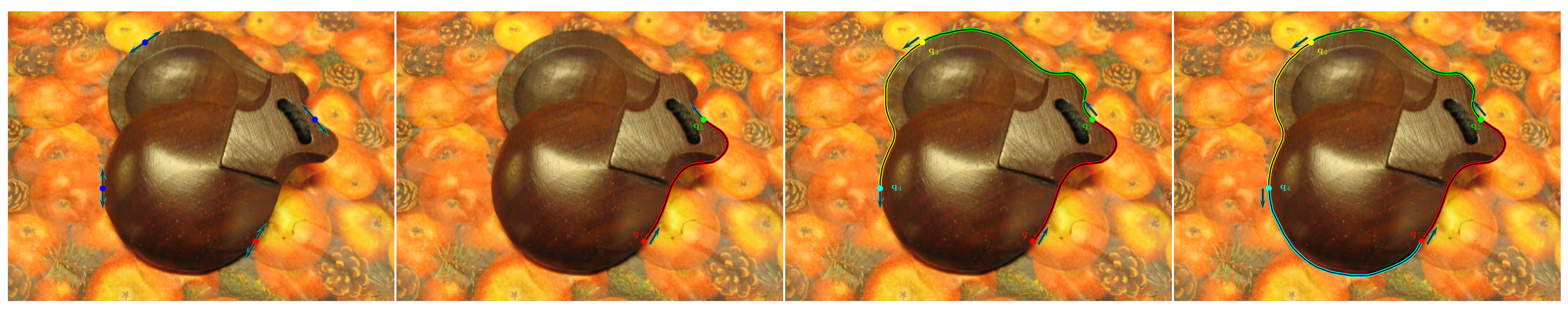

Fig. 5. Contour detection procedure. **Column 1** Initializations with a set of orientation lifted points. **Columns 2-4** Steps of the detection procedure.

Basic idea: Among a set of user provided orientation lifted points, we find the closest lifted point $\mathbf{q}_{i+1}$ to the previous $\mathbf{q}_i$ in terms of $\mathcal{P}$ based geodesic distance in the manner of matching lifted points by pair. This is done by fast marching front propagation: the closest point $\mathbf{q}_{i+1}$ is the first point reached by the monotonically advancing front which starts from $\mathbf{q}_i$. The points in each pair will be linked by minimal paths

Vessels Extraction

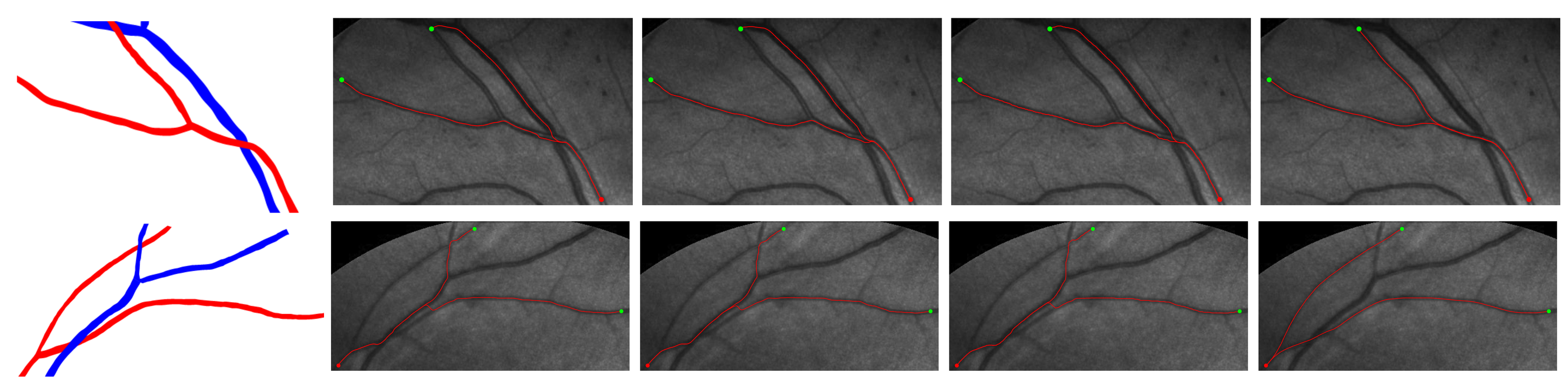

Fig. 6. **Column 1** Retinal vessel vein-artery groundtruth. **Columns 2-5**: Retinal vessel extraction by isotropic, anisotropic Riemannian metric, isotropic orientation lifted metric and Finsler elastica metric.

Conclusions

The core contributions lie at the introduction of curvature penalty to the Eikonal PDE based minimal path model. This is done by establishing the connection between the Euler elastica bending energy and the geodesic energy via a family of orientation lifted Finsler elastica metrics. Solving the Eikonal PDE with respect to the proposed Finsler elastica metric, our model thus can determine globally minimizing curves with curvature penalty between two orientation lifted points. These minimal curves are asymmetric and smooth, benefiting from the orientation lifting and curvature penalty.

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