Geodesic Distances and Curves through Isotropic and Anisotropic Heat Equations on Images and Surfaces

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1. Introduction

Q&A:

Q: What is geodesic?

A: A geodesic is the shortest path between two points. In Euclidean space, a geodesic is a **straight line** connecting two points.

Q: Why geodesics are useful?

A: Geodesics are useful for segmentation through extracting minimal paths.

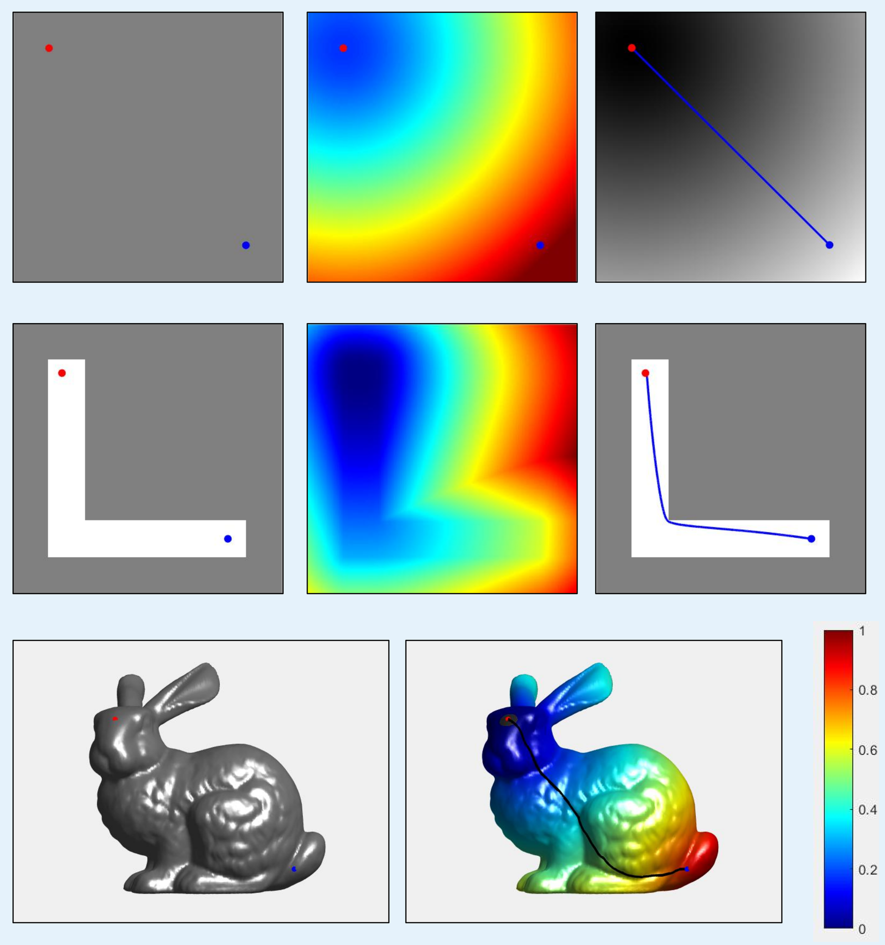
Q: How to obtain geodesics?

A: We approximate the geodesic distance by using the relation between the heat and the geodesic distance based on **Varadhan's formula**.

Q: Why we use heat method?

A: Heat method is fast and robust, and not easy to be affected by noise.

Some examples:



The shortest path connecting the source (red) point and end (blue) point in different cases. The colormap represents the distance map, the more close to the sourcepoint, the smaller the distance is.

Heat equation:

$$\partial_t u(x,t)=\alpha(x)\operatorname{div}(D\cdot\nabla u) \quad (1)$$

α stands for the thermal conductivity. In the case of a homogeneous isotropic medium, the matrix D has the form of a constant scalar times identity matrix I_d, where the scalar represents the conductivity.

Varadhan's formula:

$$\phi(p_0,p_x)=\lim_{t\to 0}\sqrt{-4t\log u_{p_0}(p_x,t)} \quad (2)$$

Varadhan's formula indicates the relation between heat kernel u and geodesic distance ϕ

Once we get an approximation of the geodesic distance ϕ, the geodesic lines γ^* between source point p_0 and other points p_x on the domain are extracted by integrating an ordinary differential equation numerically:

$$\forall s>0, \frac{d\gamma^*}{ds}=-D^{-1}\nabla\phi, \gamma^*(0)=p_x \quad (4)$$

where D is the metric tensor in the anisotropic case. For the isotropic case, $D=\alpha^2 I_d$, and Eq.(3) becomes $\frac{d\gamma^*}{ds}=-\nabla\phi$.

2. Isotropic Heat Diffusion

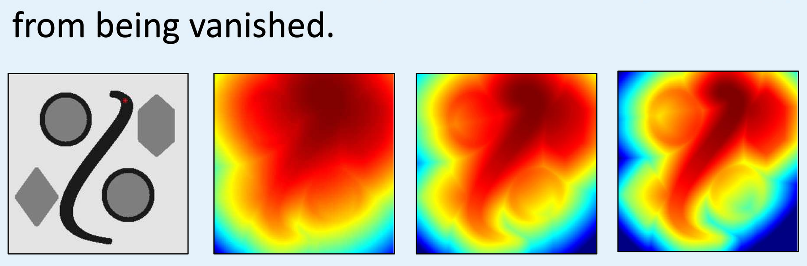
Isotropic Heat Diffusion

When the diffusion tensor D in Eq.(1) is an Identity matrix or a scalar, the heat diffusion is isotropic, we consider 4 kinds of isotropic heat diffusion:

1. Conductivity:

$$\alpha_{p_x}=\left|(1-|f(p_0)-f(p_x)|)\right|^n+\varepsilon \quad (5)$$

$f(\cdot)$ is the image, $f(p_x)$ is the graylevel of point p_x, p_0 represents the source point chosen by the user, $\varepsilon\geq 0$ is a small value that keep the conductivity α in Eq.(3) from being vanished.



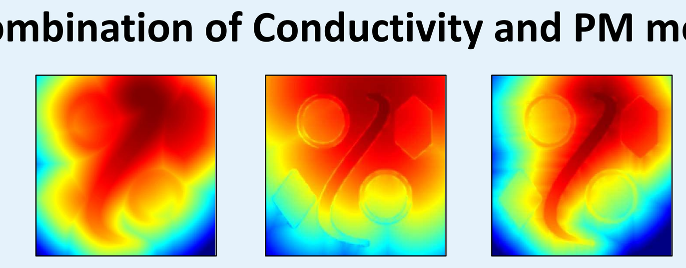
The example of how heat propagates on a synthetic image and the effect of the power n in Eq.(5). from left to right are the original image with a source point, three results with n=1,2,3 respectively.

2.Perona&Malik Model

$$D=e^{-(\|\nabla f\|/K)^2} \text{ or } D=1/(1+(\|\nabla f\|/K)^2) \quad (6)$$

K is the contrast parameter, and $\|\nabla f\|$ is the norm of the gradient of the image. Eq.(6) indicates that this P-M model inhibits heat from leaking outside a homogeneous region.

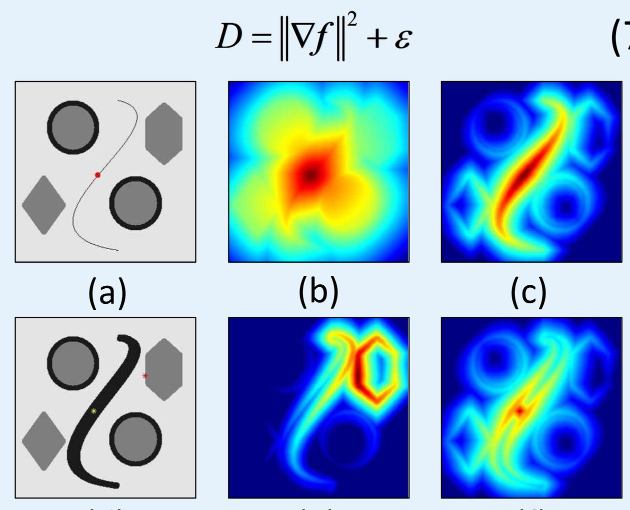
3.Combination of Conductivity and PM model



Example on the same synthetic image showing the combination of conductivity and P-M diffusivity. A source point settles on top of the curve. After the same period of time, from left to right are: the result of using the cubic form of Eq.(5), the result of PM model Eq.(6), the result of using the combination of conductivity and P-M model Eq.(10).

4. A P-M model that follows the edges:

$$D=\|\nabla f\|^2+\varepsilon \quad (7)$$



(a) (b) (c) (d) (e) (f)

Example on a synthetic image to illustrate how Eq.(7) works. The source point is placed on the very thin line in (a) and a wider curve in (d), the red point in (a) is the source point for (b) and (c). (b) is the result of diffusing on (a) by using Eq.(5), (c) is the diffusion result of (a) using Eq.(7). The red point on (d) indicates the position of the source point for (e) and the yellow point indicates the position of the source point for (f). Both (e) and (f) are the diffusion results of (d) generated by using Eq.(7) as the feature in the P-M model.

3. Anisotropic Heat Diffusion

Anisotropic Heat Diffusion

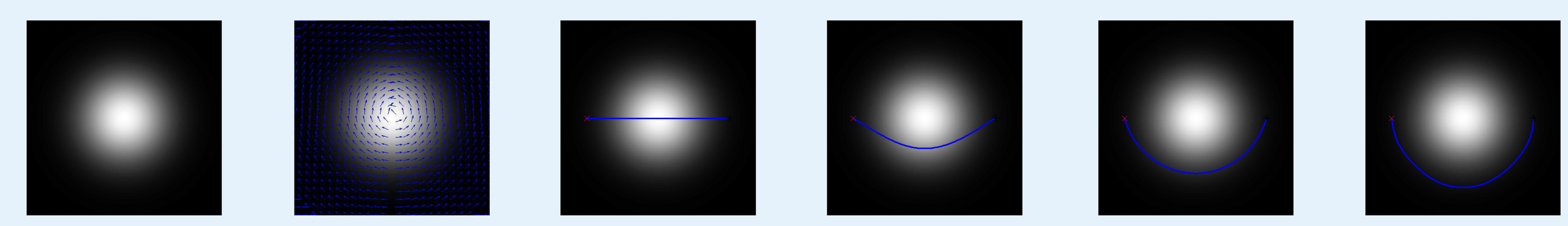
When the diffusion tensor D in Eq.(1) is a tensor, the heat diffusion becomes anisotropic. D is a tensor field of symmetric positive matrices that encodes the local orientation and anisotropy of an image.
The tensor field can be diagonalized as:

$$D_x=\lambda_1(x)e_1(x)e_1(x)^T+\lambda_2(x)e_2(x)e_2(x)^T \quad (7)$$

The normalized vector fields $e_i(x)$ are orthogonal eigenvectors of the symmetric matrix, and the $\lambda_i(x)$ are the corresponding eigenvalues, with $\lambda_2(x)\geq\lambda_1(x)>0$.
The anisotropy $A(x)$ is:

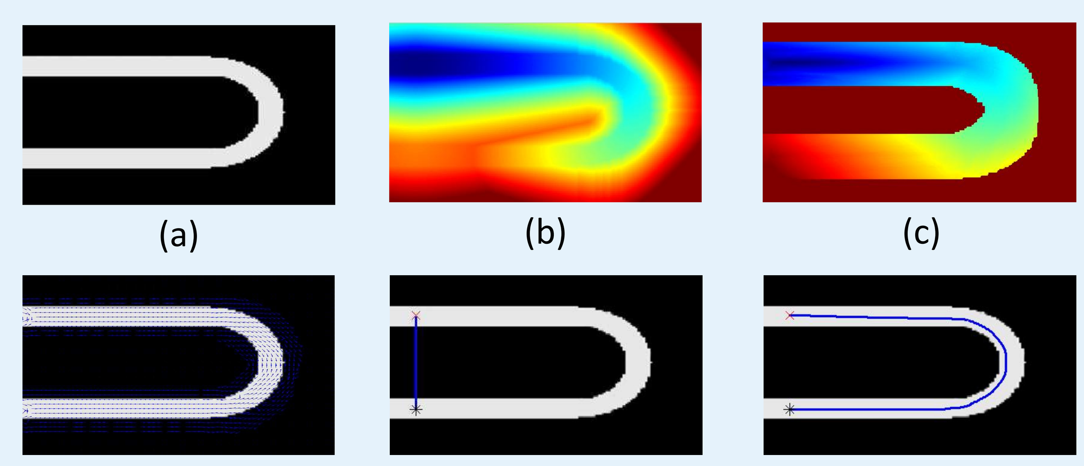
$$A(x)=\frac{\lambda_2(x)-\lambda_1(x)}{\lambda_2(x)+\lambda_1(x)} \quad (8)$$

when $\lambda_1(x)=\lambda_2(x)$, the anisotropy equals to 0, the tensor is in fact a scalar metric which makes geodesics the shortest paths according to the isotropically weighted distance.



(a) (b) (c) (d) (e) (f)

(a) a Gaussian image, (b) tensor field by gradient, (c) to (f) are the shortest paths between the two user-chosen points, the corresponding anisotropies are 0, 0.5, 0.8 and 0.99



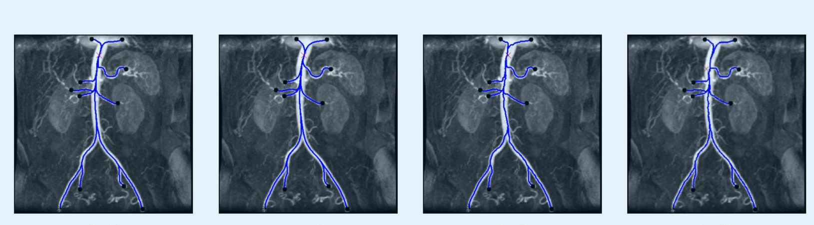
(a) (b) (c) (d) (e) (f)

Experiment on a U-tube structure: (a) original image, (d) tensor field, (b) and (c) are the distance maps obtained by isotropic and anisotropic diffusions respectively, (e) and (f) are the corresponding geodesic lines.

4. Experiments and Analysis of Different Heat Flows

(a) (b) (c) (d)

Experiment on a satellite image: from left to right, (a) displays the paths extracted by using Fast Marching, (b) is the result of isotropic heat method by using respectively Eq.(3) with n=2 and ε=0, (c) and (d) are results by using Eq(5) and Eq.(4) in the P-M model.

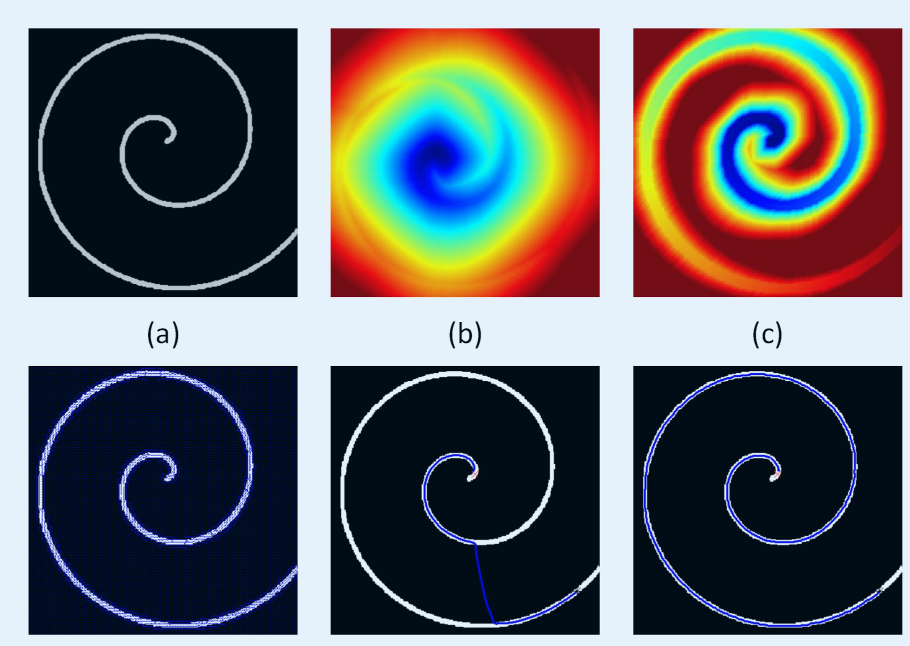


(a) (b) (c) (d)

real vessel image: the red cross is the source point, the black spots are the end points provided by the user, and the blue curves are the extracted geodesics (a) is the result by isotropic Fast Marching (b) is the result by only using the conductivity Eq.(3), (c) and (d) are the results by using the combination but with different source points.

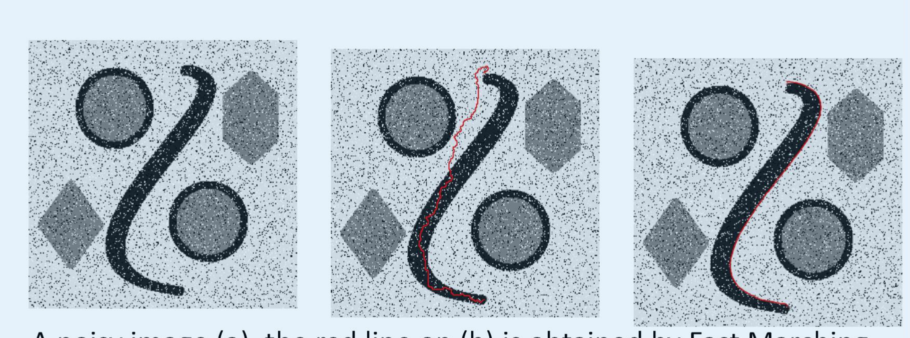
	Metrics	Advantages	disadvantages
IHF*	Conductivity	Convenient Intuitive	Complicated Scenes
	P-M Eq.(5)	Homogeneous Region	For edges
	P-M Eq.(6)	Edges & Boundaries	For regions
AHF*		Specified directions	Time Consumption

* IHF and AHF are abbreviations for the isotropic heat flow and anisotropic heat flow.

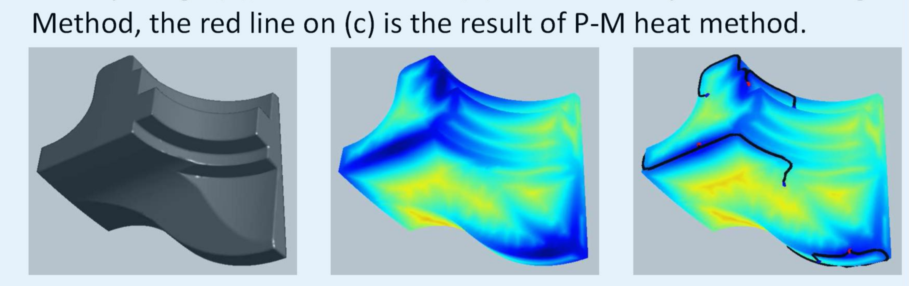


(a) (b) (c) (d) (e) (f)

(a) original spiral image; (d) tensor eld; (b) and (c) are distance maps by iso- and aniso-tropic and heat diusion; (e) and (f) are geodesic line extracted corresponding to (b) and (c).



A noisy image (a), the red line on (b) is obtained by Fast Marching Method, the red line on (c) is the result of P-M heat method.



A wedge-like surface, three points on the edges are the source points, and 10 random points are the end points. (a) the data, (b) the distance map, (c) the paths extracted along the edges.

5. Conclusion

We proposed new methods using the isotropic and anisotropic heat diffusions to get the geodesic distance and geodesic lines for image segmentation purposes. Using different kinds of diffusivity, models and tensors, the methods work well for different types of images and features of interest.

References

[1] S.R.S.Varadhan. On the behavior of the fundamental solution of the heat equation with variable coefficients. Communications on Pure and Applied Mathematics, 20(2):431–455, 1967.
[2] Keenan Crane, Clarisse Weischedel, and Max Wardetzky. Geodesics in heat: a new approach to computing distance based on heat flow. ACM Transactions on Graphics (TOG), 32(5):152, 2013.