COURSE DESCRIPTION – "*HODGE THEORY AND MODULI*"

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1. Course Description

Algebraic Geometry is the study of geometric objects defined by polynomial equations. A recurring theme in Algebraic Geometry is to study how geometric objects vary in families and how they degenerate. It turns out that frequently there is a natural space parameterizing (in a one-to-one fashion) the geometric objects within a given class (e.g. with specified topological invariants), and moreover this parameter space (aka *moduli space*) is itself an algebraic variety. The study of moduli spaces has led to numerous applications in Algebraic Geometry and nearby fields, for instance one uses moduli spaces to solve enumerative problems or to establish the existence or non-existence of geometric objects with special properties.

The purpose of the course is to survey the problem of constructing and compactifying moduli spaces for certain classes of algebraic varieties. While we will give a broad overview of various techniques involved in moduli theory (such as GIT, MMP, Hodge theory), the perspective taken here is that of Hodge theory and period maps. Among the various techniques used to construct a moduli space, Hodge theory (via the period map construction) has the distinction that it gives the richest structure to a moduli space. This leads to the application of powerful techniques of arithmetic and analytic nature (e.g. automorphic forms) to the study of moduli spaces. Essentially all that is known about the moduli space of abelian varieties and $K3$ surfaces is a consequence of the Hodge theoretic construction and the application of the associated tools. For instance, using automorphic forms, it follows relatively easy that most of the moduli spaces of principally polarized abelian varieties and $K3$ surfaces are of general type. Unfortunately, beyond abelian varieties and $K3$ surfaces (and some related examples), the applications of Hodge theory to moduli are quite limited. Time permitting, we will discuss some recent efforts to apply period maps to situations beyond the classical cases (of $K3$s and abelian varieties), e.g. to surfaces of general type with low invariants, and Calabi-Yau threefolds.

1.1. **Course Objectives.** This is an advanced course in Algebraic Geometry, which aims to give an overview of the various construction techniques in moduli theory, and to survey some recent developments at the interface between moduli and Hodge theory.

1.2. **Pre-requisites.** A basic course in Algebraic Geometry. A course in Hodge theory (e.g. [Voi02]) is recommended, but not essential. We aim for the course to be self-contained and accessible to graduate students in Algebraic Geometry and related fields.

1.3. **Recommended Texts.** We recommend Voisin [Voi02] and Carlson et al. [CMSP03] for the Hodge Theory part of our course. There is no single text covering the moduli part of the course, but Mumford et al. [MFK94], Mukai [Muk03], and Kollár [Kol13] are good general references. Our surveys ([Laz13], [Laz15], and [LZ15]) can be used as a companion to the course, and a guide to the literature.

2. Course Outline

We plan 10 lectures divided into two parts. During the first part of the course, we will review the basic theory of period maps and period domains. The basic questions here are: (1) *How is the cohomology of (smooth) varieties varying?*, and (2) *What happens, when the varieties degenerate?* Most of the material here is standard and well covered in [Voi02] and [CMSP03], but we will touch also on some newer developments such as Mumford-Tate domains ([GGK12]) and compactifications ([KU09]).

The second half will be concerned with applications of Hodge theory to moduli. We will start by reviewing the classical case of abelian varieties and $K3$ surfaces with an emphasis on applications such as the computation of the Kodaira dimension for some standard moduli spaces. We will close by discussing some more recent developments such as the study of the birational geometry of the moduli of $K3$ surfaces (cf. [LO16]), or the work of Griffiths and his collaborators on period maps for Horikawa surfaces (minimal surfaces of general type with invariants $p_g = 2, q = 0, K^2 = 2$).

Part 1 - Period maps and Period Domains

(November 15 - December 13, Tuesdays @1:30pm)

(1) Hodge structures, Variations of Hodge Structures, Period Mappings
(2) Period domains, Griffiths' Transversality, Hermitian symmetric domains, Shimura varieties.
(3) Mixed Hodge Structures and Degenerations of Hodge Structures
(4) Baily-Borel and toroidal compactifications.
(5) Properties of periods maps, Torelli Theorems

Part 2 - Moduli and Periods

(January 17 - February 14, Tuesdays @1:30pm)

(5) Introduction to moduli. Quick review of the construction and compactification problem for moduli.
(7) Basic examples: abelian varieties ($\mathcal{A}_g$), curves ($\mathcal{M}_g$), and $K3$ surfaces ($\mathcal{F}_g$)
(8) Moduli and periods for $K3$s (and related examples)
(9) Birational geometry of the moduli of $K3$ surfaces
(10) Period maps beyond the classical case

References

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[GGK12] M. Green, P. Griffiths, and M. Kerr, *Mumford-Tate groups and domains*, Annals of Mathematics Studies, vol. 183, Princeton University Press, Princeton, NJ, 2012, Their geometry and arithmetic.

[Kol13] J. Kollár, *Moduli of varieties of general type*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 131–157. MR 3184176

[KU09] K. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Annals of Mathematics Studies, vol. 169, Princeton University Press, Princeton, NJ, 2009.

[Laz13] R. Laza, *GIT and moduli with a twist*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 259–297.

[Laz15] ———, *Perspectives on moduli*, Compactifying Moduli Spaces (Lecture notes at Barcelona 2013 Summer School), 2015.

[LO16] R. Laza and K. G. O'Grady, *Birational geometry of the moduli space of quartic $K3$ surfaces*, arXiv:1607.01324, 07 2016.

[LZ15] R. Laza and Z. Zhang, *Classical period domains*, Recent Advances in Hodge Theory, Cambridge University Press, 2015.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[Muk03] S. Mukai, *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics, vol. 81, Cambridge University Press, Cambridge, 2003, Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.

[Voi02] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés, vol. 10, Société Mathématique de France, Paris, 2002.