

# Boundary Rigidity in Two Dimensions

Gunther Uhlmann\*

## Abstract

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We describe in two dimensions a connection between the scattering relation, the Hilbert transform and the geodesic X-ray transform for non-trapping Riemannian manifolds with strictly convex boundary. This connection was used to solve the boundary rigidity problem in two dimensions for simple manifolds [31]. The key point in this development is the proof that the scattering relation determines the Dirichlet-to-Neumann map for the Laplace-Beltrami operator. We also use this connection to give a characterization of the range of the geodesic X-ray transform in terms of the scattering relation for non-trapping manifolds with strictly convex boundary. and inversion formulas, on two dimensional simple manifolds, for the geodesic X-ray transform acting on scalar functions and vector fields for metrics with constant curvature and Fredholm type inversion formulas in the general case [?].

## 1 Travel Time Tomography and Boundary Rigidity

The question of determining the sound speed or index of refraction of a medium by measuring the first arrival times of waves arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. An early success of this inverse method was the estimate by Herglotz [19] and Wiechert and Zoeppritz [40] of the diameter of the Earth and the location of the mantle, crust and core. The assumption used in those papers is that the index of

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\*Department of Mathematics, University of Washington, Seattle, WA 98195, USA, and Fondation Sciences Mathématiques de Paris. email: gunther@math.washington.edu

refraction (speed of waves) depends only on the radius. A more realistic model is to assume that it depends on position. The inverse kinematic problem can be formulated mathematically as determining a Riemannian metric on a bounded domain (the Earth) given by  $ds^2 = \frac{1}{c^2(x)}dx^2$ , where  $c$  is a positive function, from the length of geodesics (travel times) joining points in the boundary.

More recently it has been realized, by measuring the travel times of seismic waves, that the inner core of the Earth might exhibit anisotropic behavior, that is the speed of waves depends also on direction there with the fast direction parallel to the Earth's spin axis [8]. Given the complications presented by modeling the Earth as an anisotropic elastic medium we consider a simpler model of anisotropy, namely that the wave speed is given by a symmetric, positive definite matrix  $g = (g_{ij})(x)$ , that is, a Riemannian metric in mathematical terms. The problem is to determine the metric from the lengths of geodesics joining points in the boundary (the surface of the Earth in the motivating example). It is useful to consider a more general and geometric formulation of the problem.

Let  $(M, g)$  be a compact Riemannian manifold with boundary  $\partial M$ . Let  $d_g(x, y)$  denote the geodesic distance between  $x$  and  $y$ , two points in the boundary. This is defined as the infimum of the length of all sufficiently smooth curves joining the two points. The function  $d_g$  measures the first arrival time of waves joining points of the boundary. The inverse problem we discuss in this section is whether we can determine the Riemannian metric  $g$  knowing  $d_g(x, y)$  for any  $x \in \partial M$ ,  $y \in \partial M$ . This problem also arose in rigidity questions in Riemannian geometry [24], [9], [16].

The metric  $g$  cannot be determined from this information alone. We have  $d_{\psi^*g} = d_g$  for any diffeomorphism  $\psi : M \rightarrow M$  that leaves the boundary pointwise fixed, i.e.,  $\psi|_{\partial M} = Id$ , where  $Id$  denotes the identity map and  $\psi^*g$  is the pull-back of the metric  $g$ . The natural question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric  $g$  and find a point  $x_0$  in  $M$  so that  $d_g(x_0, \partial M) > \sup_{x, y \in \partial M} d_g(x, y)$ . For such a metric,  $d_g$  is independent of a change of  $g$  in a neighborhood of  $x_0$ . The hemisphere of the round sphere is another example.

Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold  $(M, g)$  is *simple*, i.e.,  $M$  is simply-connected, any geodesic has no conjugate points and  $\partial M$  is strictly convex.  $\partial M$  is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point.

R. Michel conjectured in [24] that simple manifolds are boundary distance rigid, that is  $d_g$  determines  $g$  uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [16]), simple subspaces of an open hemisphere in two dimensions (see [25]), simple subspaces of symmetric spaces of constant negative curvature [3], simple two dimensional spaces of negative curvature (see [10] or [29]). There are several local and semiglobal results [36], [11], [21], [6]. We remark that simplicity of a compact manifold with boundary can be determined from the boundary distance function.

Michel's conjecture was proven in generality in [31] in two dimensions.

**Theorem 1.1.** *Let  $(M, g_i), i = 1, 2$  be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M.$$

*Then there exists a diffeomorphism  $\psi : M \rightarrow M$ ,  $\psi|_{\partial M} = Id$ , so that*

$$g_2 = \psi^* g_1.$$

In the case that both  $g_1$  and  $g_2$  are conformal to the Euclidean metric  $e$  (i.e.,  $(g_k)_{ij} = \alpha_k \delta_{ij}$ ,  $k = 1, 2$  with  $\delta_{ij}$  the Kronecker symbol), as mentioned earlier, the problem we are considering here is known in seismology as the inverse kinematic problem. In this case, it has been proven by Mukhometov in two dimensions [26] that if  $(M, g_i), i = 1, 2$  is simple and  $d_{g_1} = d_{g_2}$ , then  $g_1 = g_2$ .

More generally the same method of proof shows that if  $(M, g_i), i = 1, 2$ , are simple compact Riemannian manifolds with boundary and they are in the same conformal class then the metrics are determined by the boundary distance function. More precisely we have:

**Theorem 1.2.** *Let  $(M, g_i), i = 1, 2$  be simple compact Riemannian manifolds with boundary. Assume  $g_1 = \rho g_2$  for a positive, smooth function  $\rho$ ,  $\rho|_{\partial M} = 1$  and  $d_{g_1} = d_{g_2}$  then  $g_1 = g_2$ .*

This result and a stability estimate were proven in [26]. We remark that in this case the diffeomorphism  $\psi$  that is present in the general case must be the identity if the metrics are conformal to each other. For related results and generalizations see [5], [14], [9], [15], [28].

The proof of Theorem 1.1 involves a connection between the scattering relation and the Dirichlet-to-Neumann map (DN) associated to the Laplace-Beltrami operator. In section 2 we define the scattering relation which

quantizes the scattering operator. In Section 3 we discuss the geodesic X-ray transform and prove injective and surjectivity of the adjoint when integrating functions. In section 4 we describe the connection between the scattering relation the Hilbert transform and the geodesic X-ray transform (see Theorem 4.2). In Section 5 we discuss the main step of the proof of Theorem 1.1 which consists in showing that, under the assumptions of the theorem, we can determine the Dirichlet-to-Neumann map if we know the scattering relation. In Section 6 we use the connection indicated in section 3, to give a characterization of the range of the geodesic X-ray transform in terms of the scattering relation and we give Fredholm type inversion formulas for the geodesic X-ray transform acting on scalar functions and vector fields.

## 2 The Scattering Relation

Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [17]). A similar obstruction to the boundary rigidity problem occurs in this case with the diffeomorphism  $\psi$  equal to the identity outside a compact set. It was proven in [17] that from the wave front set of the scattering operator, one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting Alexandrova has shown for a large class of operators that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator quantized by the scattering relation [1]. The scattering relation maps the point and direction of a geodesic entering the manifold to the point and direction of exit of the geodesic. We proceed to define in more detail the scattering relation and its relation with the boundary distance function.

Let  $\nu$  denote the unit-inner normal to  $\partial M$ . We denote by  $S(M) \rightarrow M$  the unit-sphere bundle over  $M$ :

$$S(M) = \bigcup_{x \in M} S_x, \quad S_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$S(M)$  is a  $(2 \dim M - 1)$ -dimensional compact manifold with boundary, which can be written as the union  $\partial S(M) = \partial_+ S(M) \cup \partial_- S(M)$

$$\partial_{\pm} S(M) = \{(x, \xi) \in \partial S(M), \pm(\nu(x), \xi) \geq 0\}.$$

The manifold of inner vectors  $\partial_+ S(M)$  and outer vectors  $\partial_- S(M)$  intersect at the set of tangent vectors

$$\partial_0 S(M) = \{(x, \xi) \in \partial S(M), \quad (\nu(x), \xi) = 0\}.$$

Let  $(M, g)$  be an  $n$ -dimensional compact manifold with boundary. We say that  $(M, g)$  is *non-trapping* if each maximal geodesic is finite. Let  $(M, g)$  be non-trapping and the boundary  $\partial M$  is strictly convex. Denote by  $\tau(x, \xi)$  the length of the geodesic  $\gamma(x, \xi, t), t \geq 0$ , starting at the point  $x$  in the direction  $\xi \in S_x$ . This function is smooth on  $S(M) \setminus \partial_0 S(M)$ . The function  $\tau^0 = \tau|_{\partial S(M)}$  is equal zero on  $\partial_- S(M)$  and is smooth on  $\partial_+ S(M)$ . Its odd part with respect to  $\xi$

$$\tau_-^0(x, \xi) = \frac{1}{2} (\tau^0(x, \xi) - \tau^0(x, -\xi))$$

is a smooth function. We give details of these claims next.

Assume that  $(M, g)$  is embedded in  $(S, g)$  where  $S$  is a compact  $n$ -dimensional manifold without boundary.

Let  $\rho$  be a defining function for  $\partial M \hookrightarrow S$ . That is,  $\rho : S \rightarrow \mathbb{R}$  is smooth,  $M = \{x \in S : \rho(x) > 0\}$ ,  $\partial M = \{x \in S : \rho(x) = 0\}$ , and  $|\nabla \rho| = 1$  near  $\partial M$ . ( Such  $\rho$  exists: near any  $p \in \partial M$ , let  $(x', x_n)$  be semigeodesic coordinates adapted to  $\partial M$ . Then locally  $\rho(x', x_n) = x_n$ . )

Let  $\nu$  be the unit inner normal to  $\partial M$ . One has  $\nu = \nabla \rho$ . Define the unit sphere bundle

$$S(M) = \bigcup_{x \in M} S_x \quad S_x = \{\xi \in T_x M : |\xi|_g = 1\}$$

This is a  $(2n-1)$ -dimensional compact manifold with boundary  $\partial S(M)$ . We consider the inner vectors  $\partial_+ S(M)$  and the outer vectors  $\partial_- S(M)$ :

$$\partial_{\pm} S(M) = \{(x, \xi) \in \partial S(M) : \pm(\nu(x), \xi) \geq 0\}$$

The manifolds of inner and outer vectors intersect at the set of tangent vectors  $\partial_0 S(M)$ :

$$\partial_0 S(M) = \{(x, \xi) \in \partial S(M) : (\nu(x), \xi) = 0\}$$

If  $(x, \xi) \in S(M)$ , let  $\gamma(t, x, \xi)$  be the unit speed  $S$ -geodesic starting from  $x$  in the direction of  $\xi$ . Define travel time  $\tau : S(M) \rightarrow [0, \infty]$  by

$$\tau(x, \xi) = \inf\{t > 0 : \gamma(t, x, \xi) \in S \setminus M\}$$

**Definition 2.1.**  $(M, g)$  is non-trapping if  $\tau(x, \xi) < \infty$  for all  $(x, \xi) \in S(M)$ .

Remark: without the connectivity condition on  $M$ ,  $\tau$  may not be continuous.

**Definition 2.2.**  $\partial M$  is strictly convex if

$$\left. \frac{d^2}{dt^2} (\rho(\gamma(t))) \right|_{t=0} < 0$$

for any  $S$ -geodesic  $\gamma$  with  $\dot{\gamma}(0)$  tangent to  $\partial M$ .

For such  $\gamma$ ,

$$\left. \frac{d}{dt} (\rho(\gamma(t))) \right|_{t=0} = D\rho_{\gamma(t)}(\dot{\gamma}(t)|_{t=0}) = \langle \nabla \rho(\gamma(0)), \dot{\gamma}(0) \rangle = \langle \nu(\gamma(0)), \dot{\gamma}(0) \rangle = 0$$

Thus if  $\partial M$  is strictly convex and  $\dot{\gamma}(0)$  is tangent to  $\partial M$ , then  $\rho(\gamma(t)) < 0$  for  $t \in (-\epsilon, \epsilon) \setminus \{0\}$ , i.e.  $\gamma(t)$  is in  $S \setminus M$  for  $t \in (-\epsilon, \epsilon) \setminus \{0\}$ .

**Lemma 2.3.** Let  $(M, g)$  be non-trapping with  $\partial M$  strictly convex. Then  $\tau : S(M) \rightarrow \mathbb{R}$  is continuous.

*Proof.* Let  $(x_0, \xi_0) \in S(M)$  and  $t_0 = \tau(x_0, \xi_0) > 0$ . Choose  $\epsilon_0$  such that  $\inf_{t \in [0, t_0 - \epsilon_0]} \rho(\gamma(t, x_0, \xi_0)) > 0$ , then

$$\begin{aligned} & \inf_{t \in [0, t_0 - \epsilon_0]} \rho(\gamma(t, x_0, \xi_0)) > 0 \\ \implies & \inf_{t \in [0, t_0 - \epsilon_0]} \rho(\gamma(t, x, \xi)) > 0 \text{ for } (x, \xi) \text{ near } (x_0, \xi_0) \\ \implies & \gamma(t, x, \xi) \in M \text{ for } 0 \leq t \leq t_0 - \epsilon_0 \text{ and } (x, \xi) \text{ near } (x_0, \xi_0) \\ \implies & \tau(x, \xi) \geq t_0 - \epsilon_0 \text{ for } (x, \xi) \text{ near } (x_0, \xi_0). \end{aligned}$$

Furthermore, since  $\partial M$  is strictly convex, one has

$$\begin{aligned} & \rho(\gamma(t_0 + \epsilon_0, x_0, \xi_0)) < 0 \text{ for } \epsilon_0 > 0 \text{ small} \\ \implies & \rho(\gamma(t_0 + \epsilon_0, x, \xi)) < 0 \text{ for } (x, \xi) \text{ near } (x_0, \xi_0) \\ \implies & \tau(x, \xi) \leq t_0 + \epsilon_0 \text{ for } (x, \xi) \text{ near } (x_0, \xi_0) \\ \implies & \tau(x, \xi) \leq t_0 - \epsilon_0 \text{ for } (x, \xi) \text{ near } (x_0, \xi_0). \end{aligned}$$

This shows the case when  $t_0 > 0$ , and the case  $t_0 = 0$  is similar. □

**Lemma 2.4.** Let  $(M, g)$  be non-trapping and  $\partial M$  strictly convex. Then  $\tau$  is smooth on  $S(M) \setminus \partial_0 S(M)$ .

*Proof.* Let

$$h(t, x, \xi) = \rho(\gamma(t, x, \xi)) \quad , t \in \mathbb{R}, (x, \xi) \in S(M)$$

Then  $h$  is smooth. Fix  $(x_0, \xi_0) \in S(M) \setminus \partial_0 S(M)$  and let  $t_0 = \tau(x_0, \xi_0)$ . Then  $h(t_0, x_0, \xi_0) = 0$  and

$$\frac{\partial h}{\partial t}(t_0, x_0, \xi_0) = \langle \nabla \rho(r_0), \eta_0 \rangle = \langle \nu(r_0), \eta_0 \rangle$$

where  $r_0 = \gamma(t_0, x_0, \xi_0)$ ,  $\eta_0 = \dot{\gamma}(t_0, x_0, \xi_0)$ . Now if  $\eta_0$  were tangent to  $\partial M$ , then  $\gamma|_{(t_0-\epsilon, t_0+\epsilon)}$  would lie outside of  $M$ , a contradiction. Thus  $\frac{\partial h}{\partial t}(t_0, x_0, \xi_0) \neq 0$ , and by the implicit function theorem there exists  $t = t(x, \xi)$  which is smooth near  $(x_0, \xi_0)$  such that

$$h(t, x, \xi) = 0 \iff t = t(x, \xi) \text{ for } (t, x, \xi) \text{ near } (t_0, x_0, \xi_0)$$

Since  $h(\tau(x, \xi), x, \xi) = 0$  and  $\tau$  is continuous, we get  $\tau = t$ .  $\square$

It is easy to see that  $\tau$  is not smooth on  $\partial_0 S(M)$ . We define

$$\tau^0(x, \xi) = \begin{cases} \tau(x, \xi) & (x, \xi) \in \partial_+ S(M) \\ -\tau(x, -\xi) & (x, \xi) \in \partial_- S(M) \end{cases}$$

**Lemma 2.5.** *Suppose  $(M, g)$  is non-trapping and  $\partial M$  is strictly convex. Then  $\tau^0 : \partial S(M) \rightarrow \mathbb{R}$  is smooth.*

*Proof.*  $\tau^0$  is continuous on  $\partial SM$  by earlier lemma. As before, we define  $h(t, x, \xi) = \rho(\gamma(t, x, \xi))$ . Then

$$\begin{aligned} h(0, x, \xi) &= 0 \\ \frac{\partial h}{\partial t}(0, x, \xi) &= \langle \nu(x), \xi \rangle \\ \frac{\partial^2 h}{\partial t^2}(0, x, \xi) &= \left. \frac{d^2}{dt^2} \right|_{t=0} (\rho(\gamma(t, x, \xi))) = c(x, \xi) \end{aligned}$$

where  $c$  is smooth and  $c < 0$  by strict convexity. Then

$$h(t, x, \xi) = \langle \nu(x), \xi \rangle t + \frac{1}{2}c(x, \xi)t^2 + R(t, x, \xi)t^3$$

with  $R$  smooth. Fix  $(x_0, \xi_0) \in \partial_0 S(M)$  and  $t_0 = 0$ . Define

$$F(t, x, \xi) = \langle \nu(x), \xi \rangle t + \frac{1}{2}c(x, \xi)t^2 + R(t, x, \xi)t^2 = \frac{\rho(\gamma(t, x, \xi))}{t}$$

Then  $\frac{\partial F}{\partial t}(t_0, x_0, \xi_0) = 0$ , so there is a smooth function  $t = t(x, \xi)$  such that

$$F(t, x, \xi) = 0 \iff t = t(x, \xi) \text{ for } (t, x, \xi) \text{ near } (t_0, x_0, \xi_0)$$

Since  $F(\tau^0(x, \xi), x, \xi) = 0$ , one sees that  $\tau^0$  is smooth.  $\square$

Now we are ready to define rigorously the scattering relation.

**Definition 2.6.** *Let  $(M, g)$  be non-trapping with strictly convex boundary. The scattering relation  $\alpha : \partial S(M) \rightarrow \partial S(M)$  is defined by*

$$\alpha(x, \xi) = (\gamma(x, \xi, 2\tau_-^0(x, \xi)), \dot{\gamma}(x, \xi, 2\tau_-^0(x, \xi))).$$

The scattering relation is a diffeomorphism  $\partial S(M) \rightarrow \partial S(M)$ . Notice that  $\alpha|_{\partial_+ S(M)} : \partial_+ S(M) \rightarrow \partial_- S(M)$ ,  $\alpha|_{\partial_- S(M)} : \partial_- S(M) \rightarrow \partial_+ S(M)$  are diffeomorphisms as well. Obviously,  $\alpha$  is an involution,  $\alpha^2 = id$  and  $\partial_0 S(M)$  is the hypersurface of its fixed points,  $\alpha(x, \xi) = (x, \xi)$ ,  $(x, \xi) \in \partial_0 S(M)$ .

A natural inverse problem is whether the scattering relation determines the metric  $g$  up to an isometry which is the identity on the boundary. This information takes into account all the travel times not just the first arrivals.

In the case that  $(M, g)$  is a simple manifold, and we know the metric at the boundary (and this is determined if  $d_g$  is known, see [25]), knowing the scattering relation is equivalent to knowing the boundary distance function ([24]).

We introduce the operators of even and odd continuation with respect to  $\alpha$ :

$$\begin{aligned} A_\pm w(x, \xi) &= w(x, \xi), & (x, \xi) \in \partial_+ S(M), \\ A_\pm w(x, \xi) &= \pm (\alpha^* w)(x, \xi), & (x, \xi) \in \partial_- S(M). \end{aligned}$$

We will examine next the boundness properties of  $A_-, A_+$ . The standard measures that we will use are defined below,

$$\begin{aligned} d\Sigma^{2n-1} &= dV^n \wedge dS_x \\ d\Sigma^{2n-2} &= dV^{n-1} \wedge dS_x \end{aligned}$$

where  $dV^n$  (resp.  $dV^{n-1}$ ) is the volume form of  $M$  (resp.  $\partial M$ ), and  $dS = \sqrt{\det g(x)} dS_x$  where  $dS_x$  is the Euclidean volume form of  $S_x$  in  $T_x M$ . For  $(x, \xi) \in \partial S(M)$ , let  $\mu(x, \xi) = \langle \nu(x), \xi \rangle$  and  $L_\mu^2(\partial_+ S(M))$  is the space of functions on  $\partial_+ S(M)$  with inner product

$$(u, v)_{L_\mu^2(\partial_+ S(M))} = \int_{\partial_+ S(M)} uv \mu d\Sigma^{2n-2}$$

**Lemma 2.7.**  $A_{\pm} : L_{\mu}^2(\partial_+ S(M)) \rightarrow L_{|\mu|}^2(\partial S(M))$  are bounded.

*Proof.*

$$\begin{aligned} \|A_{\pm} w\|_{L_{|\mu|}^2(\partial S(M))}^2 &= \int_{\partial_+ S(M)} w^2 \mu d\Sigma^{2n-2} + \int_{\partial_- S(M)} (\alpha^* w)^2 (-\mu d\Sigma^{2n-2}) \\ &= \int_{\partial_+ S(M)} w^2 \mu d\Sigma^{2n-2} + \int_{\partial_+ S(M)} w^2 \alpha^* (-\mu d\Sigma^{2n-2}) \end{aligned}$$

where  $\alpha : \partial_+ S(M) \rightarrow \partial_- S(M)$  is a diffeomorphism. Thus it is enough to show that

$$\alpha^* (-\mu d\Sigma^{2n-2}) = \mu d\Sigma^{2n-2}$$

Let  $w \in C^\infty(\partial_+ S(M))$ . Then

$$\int_{\partial_+ S(M)} w \tau \mu d\Sigma^{2n-2} = \int_{\partial_+ S(M)} \int_0^{\tau(x, \xi)} w_\psi(\varphi_t(x, \xi)) \mu dt d\Sigma^{2n-2} = \int_{S(M)} w_\psi d\Sigma^{2n-1}$$

Set  $\tilde{u}(x, \xi) = u(x, -\xi)$  for  $u \in C^\infty(S(M))$ , one has

$$\begin{aligned} \int_{S(M)} w_\psi d\Sigma^{2n-1} &= \int_{S(M)} \tilde{w}_\psi d\Sigma^{2n-1} \\ &= \int_{\partial_- S(M)} \int_0^{\tau(y, -\eta)} \tilde{w}_\psi(\varphi_t(y, -\eta)) (-\mu) dt d\Sigma^{2n-2} \\ &= \int_{\partial_- S(M)} \int_0^{\tau(y, -\eta)} w(\alpha(y, \eta)) (-\mu) dt d\Sigma^{2n-2} \\ &= \int_{\partial_+ S(M)} w \tau \alpha^* (-\mu d\Sigma^{2n-2}) \end{aligned}$$

Varying  $w$  shows that  $\alpha^* (-\mu d\Sigma^{2n-2}) = \mu d\Sigma^{2n-2}$  on  $\partial_+ S(M) \setminus \partial_0 S(M)$ .  $\square$

The adjoint  $A_{\pm}^* : L_{|\mu|}^2(\partial S(M)) \rightarrow L_{\mu}^2(\partial_+ S(M))$  satisfies

$$\begin{aligned} (A_{\pm} w, u)_{L_{|\mu|}^2(\partial S(M))} &= \int_{\partial_+ S(M)} w u \mu d\Sigma^{2n-2} \pm \int_{\partial_- S(M)} (w \circ \alpha) u (-\mu d\Sigma^{2n-2}) \\ &= \int_{\partial_+ S(M)} w (u \pm u \circ \alpha) \mu d\Sigma^{2n-2} \end{aligned}$$

so  $A_{\pm}^* u = (u \pm u \circ \alpha)|_{\partial_+ S(M)}$ .

### 3 The geodesic X-ray transform

The X-ray transform integrates a function along lines. Radon found in 1917 an inversion formula in two dimensions to determine a function knowing the X-ray transform. This formula is non-local in the sense that in order to find the function at a point  $x$  one needs to know the integral of the function along lines far from the point. Radon's inversion formula has been implemented numerically using the filtered backprojection algorithm which is used today in CT scans. Another important transform in medical imaging and other applications is the Doppler transform which integrates a vector field along lines. The motivation is ultrasound Doppler tomography. It is known that blood flow is irregular and faster around tumor tissue than in normal tissue and Doppler tomography attempts to reconstruct the blood flow pattern. Mathematically the problem is to what extent a vector field is determined from its integral along lines.

In these notes we consider the case of integrating functions and vector fields along geodesics of a Riemannian metric. This arises in geophysics since the ray paths are no longer straight lines. We obtain inversion formulas for the constant curvature case and Fredholm type formulas in general which are *non-local*. We define next the geodesic X-ray transform for any compact Riemannian manifold  $(M, g)$  with boundary of any dimension.

Let  $\varphi_t$  be the geodesic flow and  $X = \frac{d}{dt}\varphi_t|_{t=0}$  be the geodesic vector field. Let  $u^f$  be the solution of the boundary value problem

$$Xu = -f, \quad u|_{\partial_- S(M)} = 0,$$

which can be written as

$$u^f(x, \xi) = \int_0^{\tau(x, \xi)} f(\varphi_t(x, \xi)) dt, \quad (x, \xi) \in S(M).$$

In particular

$$X\tau = -1.$$

The trace

$$If = u^f|_{\partial_+ S(M)}$$

is called *the geodesic X-ray transform* of the function  $f$ . If the manifold  $(M, g)$  is non-trapping has a strictly convex boundary the operator  $I : C^\infty(S(M)) \rightarrow C^\infty(\partial_+ S(M))$ .

Clearly a function  $f$  is not determined by its geodesic X-ray transform alone, since it depends on more variables than  $If$ . We consider the geodesic X-ray transform acting on symmetric tensor fields.

We denote by  $f_m(x, \xi)$  an homogeneous polynomial of degree  $m$  with respect to  $\xi$ , induced by the symmetric tensor field  $f$  on  $(M, g)$  of  $m$  degree :

$$f_m(x, \xi) = f_{i_1 \dots i_m}(x) \xi^{i_1} \dots \xi^{i_m}.$$

The operator  $I_m$ , defined by

$$I_m f = If_m$$

is called *the geodesic X-ray transform* of the symmetric tensor field. If the manifold  $(M, g)$  is non-trapping and the boundary  $\partial M$  is strictly convex  $I_m : C^\infty(M, S_m(M)) \rightarrow C^\infty(\partial_+ S(M))$ , where  $S_m(M)$  denotes the bundle of symmetric tensors over  $(M, g)$ . It is known that any symmetric smooth enough tensor field  $f$  may be decomposed in a potential and solenoidal part [34]:

$$f = dp + h, \quad p|_{\partial M} = 0, \quad \delta h = 0,$$

where  $\delta$  notes the divergence and  $d = \sigma \nabla$  is the symmetric part of covariant derivative. It is easy to see that the geodesic X-ray transform of the potential part  $dp$  is zero. We denote by  $C_{sol}^\infty(M, S_m(M))$  the space of smooth solenoidal symmetric tensor fields.

We will consider in these notes only the case of the geodesic X-ray transform acting on functions independent of  $\xi$  and the geodesic X-ray transform acting on vector fields which, following the notation above, are denoted by  $I_0$  and  $I_1$  respectively. It is known that  $I_0$  is injective on simple manifolds [26] and that  $I_1$  is injective acting on solenoidal vector fields on simple manifolds [An]. We mention also that the transform  $I_2$  arises in the linearization of the boundary rigidity problem (see [34]). We define  $\psi : S(M) \rightarrow \partial_- S(M)$  by

$$\psi(x, \xi) = \varphi_{-\tau(x, -\xi)}(x, \xi), \quad (x, \xi) \in S(M).$$

So,  $\varphi$  is the end point which maps the vector  $(x, \xi)$  along the geodesic  $\gamma(x, \xi, t)$  in the back direction into an incoming vector. The solution of the boundary value problem for the transport equation

$$Xu = 0, \quad u|_{\partial_+ S(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \psi.$$

To study the mapping properties of  $I$ , we recall the Santalo's formula

$$\int_{S(M)} f(x, \xi) d\Sigma^{2n-1} = \int_{\partial_+ S(M)} \int_0^{\tau(x, \xi)} f(\varphi_t(x, \xi)) \langle \nu(x), \xi \rangle dt d\Sigma^{2n-2}$$

This is a change-of-variable formula which is valid when  $(M, g)$  is non-trapping with smooth boundary, and  $f \in C(S(M))$ .

**Lemma 3.1.**  $I : L^2(M) \rightarrow L^2_\mu(\partial_+ S(M))$  is bounded.

*Proof.* Since  $(M, g)$  is non-trapping, we have

$$\begin{aligned} |If(x, \xi)|^2 &= \left| \int_0^{\tau(x, \xi)} f(\varphi_t(x, \xi)) dt \right|^2 \\ &\leq C \int_0^{\tau(x, \xi)} f^2(\varphi_t(x, \xi)) dt \end{aligned}$$

so

$$\begin{aligned} \int_{\partial_+ S(M)} |If(x, \xi)|^2 \mu d\Sigma^{2n-2} &\leq C \int_{\partial_+ S(M)} \int_0^{\tau(x, \xi)} f^2(\varphi_t(x, \xi)) \mu dt d\Sigma^{2n-2} \\ &= C \int_{S(M)} f^2 d\Sigma^{2n-1} \\ &= C \int_M f^2 dV \end{aligned}$$

where we have used the Santalo's formula.  $\square$

The adjoint  $I^*$  is bounded  $L^2_\mu(\partial_+ S(M)) \rightarrow L^2(M)$ . For  $f \in C^\infty(M)$ ,  $w \in C^\infty(\partial_+ S(M))$ ,

$$\begin{aligned} (If, w)_{L^2_\mu(\partial_+ S(M))} &= \int_{\partial_+ S(M)} \int_0^{\tau(x, \xi)} f(\varphi_t(x, \xi)) w_\psi(\varphi_t(x, \xi)) \mu dt d\Sigma^{2n-2} \\ &= \int_{S(M)} f w_\psi d\Sigma^{2n-1} \\ &= \int_{S(M)} f(x) \left( \int_{S_x} w_\psi(x, \xi) dS_x(\xi) \right) dV^n(x) \end{aligned}$$

so

$$I^* w(x) = \int_{S_x} w_\psi(x, \xi) dS_x$$

The adjoint of the operator  $I_m$  is the bounded operator  $I_m^* : L_\mu^2(\partial_+ S(M)) \rightarrow L^2(M, S_m(M))$  which is given by

$$(I_m^* w)^{i_1 \dots i_m}(x) = \int_{S_x} w_\psi(x, \xi) \xi^{i_1} \dots \xi^{i_m} dS_x.$$

The Hilbert space  $L^2(M, S_m(M))$  may be considered as subspace of  $L^2(S(M))$  of homogeneous polynomials with respect to  $\xi$  of  $m$  degree. The field  $I_m^* w$  is solenoidal in the sense of the theory of distributions. Notice, that adjoint of the bounded operator  $I : L^2(S(M)) \rightarrow L_\mu^2(\partial_+ S(M))$  is given by

$$I^* w = w_\psi.$$

We also remark that by the fundamental theorem of calculus we have

$$IXf = (f \circ \alpha - f)|_{\partial_+ S(M)} = -A_-^* f^0, f^0 = f|_{\partial S(M)}. \quad (1)$$

The space  $C_\alpha^\infty(\partial_+ S(M))$  is defined by

$$C_\alpha^\infty(\partial_+ S(M)) = \{w \in C^\infty(\partial_+ S(M)) : w_\psi \in C^\infty(S(M))\}.$$

In [31] the following characterization of the space of smooth solutions of the transport equation was given

**Lemma 3.2.**

$$C_\alpha^\infty(\partial_+ S(M)) = \{w \in C^\infty(\partial_+ S(M)) : A_+ w \in C^\infty(\partial S(M))\}.$$

Then  $I^* w \in C^\infty(M)$  whenever  $w \in C_\alpha^\infty(\partial_+ S(M))$ .

### 3.1 Injectivity of the Geodesic Ray Transform

In this section we consider the Pestov identity, which is the basic energy identity that has been used since the work of Mukhometov [26] in most injectivity proofs of ray transforms in the absence of real-analyticity or special symmetries. The Pestov identity often appears in a somewhat ad hoc way, but in [30] a new point of view was given which makes its derivation more transparent.

Since  $M$  is assumed oriented there is a circle action on the fibers of  $SM$  with infinitesimal generator  $V$  called the *vertical vector field*. It is possible to complete the pair  $X, V$  to a global frame of  $T(SM)$  by considering the vector field  $X_\perp := [X, V]$ . There are two additional structure equations given by

$X = [V, X_\perp]$  and  $[X, X_\perp] = -KV$  where  $K$  is the Gaussian curvature of the surface. Using this frame we can define a Riemannian metric on  $SM$  by declaring  $\{X, X_\perp, V\}$  to be an orthonormal basis and the volume form of this metric will be denoted by  $d\Sigma^3$ . The fact that  $\{X, X_\perp, V\}$  are orthonormal together with the commutator formulas implies that the Lie derivative of  $d\Sigma^3$  along the three vector fields vanishes.

We consider the ray transform on functions. The first step is to recast the injectivity problem as a uniqueness question for the partial differential operator  $P$  on  $SM$  where

$$P := VX.$$

This involves a standard reduction to the transport equation.

**Proposition 3.1.** *Let  $(M, g)$  be a compact oriented nontrapping surface with strictly convex smooth boundary. The following statements are equivalent.*

- (a) *The ray transform  $I : C^\infty(M) \rightarrow C(\partial_+(SM))$  is injective.*
- (b) *Any smooth solution of  $Pu = 0$  in  $SM$  with  $u|_{\partial(SM)} = 0$  is identically zero.*

*Proof.* Assume that the ray transform is injective, and let  $u \in C^\infty(SM)$  solve  $Pu = 0$  in  $SM$  with  $u|_{\partial(SM)} = 0$ . This implies that  $Xu = -f$  in  $SM$  for some smooth  $f$  only depending on  $x$ , and we have  $0 = u|_{\partial_+(SM)} = If$ . Since  $I$  is injective one has  $f = 0$  and thus  $Xu = 0$ , which implies  $u = 0$  by the boundary condition.

Conversely, assume that the only smooth solution of  $Pu = 0$  in  $SM$  which vanishes on  $\partial(SM)$  is zero. Let  $f \in C^\infty(M)$  be a function with  $If = 0$ , and define the function

$$u(x, v) := \int_0^{\tau(x, v)} f(\gamma(t, x, v)) dt, \quad (x, v) \in SM.$$

This function satisfies the transport equation  $Xu = -f$  in  $SM$  and  $u|_{\partial(SM)} = 0$  since  $If = 0$ , and also  $u \in C^\infty(SM)$ . Since  $f$  only depends on  $x$  we have  $Vf = 0$ , and consequently  $Pu = 0$  in  $SM$  and  $u|_{\partial(SM)} = 0$ . It follows that  $u = 0$  and also  $f = -Xu = 0$ .  $\square$

We now focus on proving a uniqueness statement for solutions of  $Pu = 0$  in  $SM$ . For this it is convenient to express  $P$  in terms of its self-adjoint and skew-adjoint parts in the  $L^2(SM)$  inner product as

$$P = A + iB, \quad A := \frac{P + P^*}{2}, \quad B := \frac{P - P^*}{2i}.$$

Here the formal adjoint  $P^*$  of  $P$  is given by

$$P^* := XV.$$

In fact, if  $u \in C^\infty(SM)$  with  $u|_{\partial(SM)} = 0$ , then

$$\begin{aligned} \|Pu\|^2 &= ((A + iB)u, (A + iB)u) = \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned} \tag{2}$$

This computation suggests to study the commutator  $i[A, B]$ . We note that the argument just presented is typical in the proof of  $L^2$  Carleman estimates [20].

By the definition of  $A$  and  $B$  it easily follows that  $i[A, B] = \frac{1}{2}[P^*, P]$ . By the commutation formulas for  $X$ ,  $X_\perp$  and  $V$ , this commutator may be expressed as

$$\begin{aligned} [P^*, P] &= XVVX - VXXV = VXVX + X_\perp VX - VXVX - VXX_\perp \\ &= V[X_\perp, X] - X^2 \\ &= -X^2 + VKV. \end{aligned}$$

Consequently

$$([P^*, P]u, u) = \|Xu\|^2 - (KVu, Vu).$$

If the curvature  $K$  is nonpositive, then  $[P^*, P]$  is positive semidefinite. More generally, one can try to use the other positive terms in (2). Note that

$$\|Au\|^2 + \|Bu\|^2 = \frac{1}{2}(\|Pu\|^2 + \|P^*u\|^2).$$

The identity (2) may then be expressed as

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u).$$

Moving the term  $\|P^*u\|^2$  to the other side, we have now proved the version of the Pestov identity which is most suited for our purposes. The main point in this proof was that the Pestov identity boils down to a standard  $L^2$  estimate based on separating the self-adjoint and skew-adjoint parts of  $P$  and on computing one commutator,  $[P^*, P]$ .

**Proposition 3.2.** *If  $(M, g)$  is a compact oriented surface with smooth boundary, then*

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 - \|VXu\|^2 = 0$$

for any  $u \in C^\infty(SM)$  with  $u|_{\partial(SM)} = 0$ .

It is well known (cf. proof of [13, Proposition 7.2]) that on a simple surface, one has

$$\|XVu\|^2 - (KVu, Vu) \geq 0, \quad u \in C^\infty(SM), \quad u|_{\partial(SM)} = 0.$$

Also, if  $Xu = -f$  where  $f = f_0 + f_1 + f_{-1}$  is the sum of a 0-form and 1-form, we have

$$\|Xu\|^2 - \|VXu\|^2 = \|f_0\|^2 \geq 0.$$

This term may be negative, and the Pestov identity may not give useful information unless there is some extra positivity like a curvature bound.

In the scalar case the following result holds on the solvability of  $I_m^*$ ,  $m = 0, 1$  [31].

**Theorem 3.3.** *Let  $(M, g)$  be a simple, compact Riemannian manifold with boundary. Then the operator  $I_0^* : C_\alpha^\infty(\partial_+ S(M)) \rightarrow C^\infty(M)$  is onto.*

*Proof.* The proof follows by showing that  $I_0^* I_0$  is an elliptic pseudodifferential operator on  $S$  therefore Fredholm and with close range. The surjectivity of  $I_0^*$  follows then since  $I_0$  is injective.

It is easy to see, that

$$(I_0^* I_0 f)(x) = \int_{\Omega_x} d\Omega_x \int_{-\tau(x, -\xi)}^{\tau(x, \xi)} f(\gamma(x, \xi, t)) dt = 2 \int_{\Omega_x} d\Omega_x \int_0^{\tau(x, \xi)} f(\gamma(x, \xi, t)) dt. \quad (3)$$

Before we continue we make a remark concerning notation. We have used up to know the notation  $\gamma(x, \xi, t)$  for a geodesic. But it is known, that a geodesic depends smoothly on the point  $x$  and vector  $\xi t \in T_x(M)$ . Therefore in what follows we will also use sometimes the notation  $\gamma(x, \xi t)$  for a geodesic. Since the manifold  $M$  is simple and any small enough neighborhood  $U$  (in  $(S, g)$ ) is also simple (an open domain is simple if its closure is simple). For any point  $x \in U$  there is an open domain  $D_x^U \subset T_x(U)$  such that exponential map  $exp_x : D_x^U \rightarrow U$ ,  $exp_x \eta = \gamma(x, \eta)$  is a diffeomorphism onto  $U$ . Let  $D_x$ ,  $x \in M$  be the inverse image of  $M$ , then  $exp_x(D_x) = M$  and  $exp_x|_{D_x} : D_x \rightarrow M$  is a diffeomorphism.

Now we change variables in (3.1),  $y = \gamma(x, \xi t)$ . Then  $t = d_g(x, y)$  and

$$(I^* I f)(x) = \int_M K(x, y) f(y) dy,$$

where

$$K(x, y) = 2 \frac{\det(\exp_x^{-1})'(x, y) \sqrt{\det g(x)}}{d_g^{n-1}(x, y)}.$$

Notice, that since

$$\gamma(x, \eta) = x + \eta + O(|\eta|^2), \quad (4)$$

it follows, that the Jacobian matrix of the exponential map is 1 at 0, and then  $\det(\exp_x^{-1})'(x, x) = 1/\det(\exp_x)'(x, 0) = 1$ . From (3) we also conclude that

$$d^2(x, y) = G_{ij}(x, y) (x - y)^i (x - y)^j, \quad G_{ij}(x, x) = g_{ij}(x), \quad G_{ij} \in C^\infty(M \times M)$$

Therefore the kernel of  $I^*I$  can be written in the form

$$K(x, y) = \frac{2 \det(\exp_x^{-1})'(x, y) \sqrt{\det g(x)}}{\left(G_{ij}(x, y) (x - y)^i (x - y)^j\right)^{(n-1)/2}}.$$

Thus the kernel  $K$  has at the diagonal  $x = y$  a singularity of type  $|x - y|^{-n+1}$ . The kernel

$$K_0(x, y) = \frac{2\sqrt{\det g(x)}}{\left(g_{ij}(x) (x - y)^i (x - y)^j\right)^{(n-1)/2}}$$

has the same singularity. Clearly, the difference  $K - K_0$  has a singularity of type  $|x - y|^{-n+2}$ . Therefore the principal symbols of both operators coincide. The principal symbol of the integral operator, corresponding to the kernel  $K_0$  coincide with its full symbol and is easily calculated. As a result

$$\sigma(I^*I)(x, \xi) = 2\sqrt{\det g(x)} \int \frac{e^{-i(y, \xi)}}{(g_{ij}(x) y^i y^j)^{(n-1)/2}} dy = c_n |\xi|^{-1}.$$

□

The analog result for vector fields was proven in [12].

**Theorem 3.4.** *Let  $(M, g)$  be a simple, compact Riemannian manifold with boundary. Then for any field  $v \in C_{sol}^\infty(M, T(M))$  there exists a function  $w \in C_\alpha^\infty(\partial_+ S(M))$  such*

$$v = I_1^* w$$

We remark that in [30] it was shown that  $I_m$  is injectivity on solenoidal tensors for simple manifolds. The case  $m = 2$  had been settled in [35].

## 4 the Hilbert Transform

We recall first the definition of the Hilbert transform on the unit disc  $\partial\mathbb{D}$ . Writing  $x_1 + ix_2 = (x_1, x_2)$ , we get  $\bar{\partial}u = \nabla u = (\partial_1 u, \partial_2 u)$ , and  $-i\bar{\partial}v = \nabla_{\perp} v = (\partial_2 v, -\partial_1 v)$ . Thus  $u$  and  $v$  are conjugate harmonic iff

$$\nabla u = \nabla_{\perp} v, \quad \nabla v = -\nabla_{\perp} u$$

This is an invariant formulation, and can be used to define conjugate harmonic functions in  $(T_x M, g(x)) \simeq (\mathbb{R}^2, e)$  by the following lemma ( then  $\nabla_{\perp} = \epsilon \nabla$  ):

**Lemma 4.1.** *Let  $M$  be a 2D oriented manifold. Then  $\exists$  a unique 2-tensor field  $\epsilon$  ( "multiplication by  $-i$ " ) such that  $\{v, -\epsilon v\}$  is a positive orthonormal basis of  $T_x M$  whenever  $v \in T_x M$  with  $|v| = 1$ . It holds that*

$$\langle \epsilon v, \epsilon w \rangle = \langle v, w \rangle, \quad \langle \epsilon v, w \rangle = -\langle v, \epsilon w \rangle$$

The Hilbert transform on  $\partial\mathbb{D}$  is

$$Hf(z) = P.V. \int_{\partial\mathbb{D}} \frac{1 + Re(z\bar{w})}{-Re(iz\bar{w})} f(w) dm(w) \quad \text{where } dm(e^{i\theta}) = \frac{1}{2\pi} d\theta$$

Write  $x_1 + ix_2 = (x_1, x_2)$ , then  $z \cdot w = Re(z\bar{w})$  and  $-iz = \epsilon z = z_{\perp}$ , so

$$Hf(z) = P.V. \int_{\partial\mathbb{D}} \frac{1 + z \cdot w}{z_{\perp} \cdot w} f(w) dm(w)$$

Let now  $u \in C^{\infty}(S(M))$ . The fiberwise Hilbert transform is defined by

$$Hu(x, \xi) = P.V. \frac{1}{2\pi} \int_{S_x} \frac{1 + \langle \xi, \eta \rangle}{\langle \xi_{\perp}, \eta \rangle} u(x, \eta) dS_x(\eta) \quad \xi \in S_x. \quad (5)$$

Here  $\perp$  means a  $90^\circ$  degree rotation. In coordinates  $(\xi_{\perp})_i = \epsilon_{ij} \xi^j$ , where

$$\epsilon = \sqrt{\det g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform  $H$  transforms even (respectively odd) functions with respect to  $\xi$  to even (respectively odd) ones. If  $H_+$  (respectively  $H_-$ ) is the even (respectively odd) part of the operator  $H$ :

$$H_+ u(x, \xi) = \frac{1}{2\pi} \int_{S_x} \frac{(\xi, \eta)}{(\xi_{\perp}, \eta)} u(x, \eta) dS_x(\eta),$$

$$Hu_-(x, \xi) = \frac{1}{2\pi} \int_{S_x} \frac{1}{(\xi_\perp, \eta)} u(x, \eta) dS_x(\eta)$$

and  $u_+, u_-$  are the even and odd parts of the function  $u$ , then  $H_+u = Hu_+, H_-u = Hu_-$ . The above integrals are understood in the principal value sense.

We have that  $X_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$ , where  $\nabla_\perp = \varepsilon \nabla$  and  $\nabla$  is the covariant derivative with respect to the metric  $g$ . The following commutator formula for the geodesic vector field and the Hilbert transform, is a crucial ingredient in the proofs of the main theorems surveyed in these notes (see [31]).

**Theorem 4.2.** *Let  $(M, g)$  be a two dimensional Riemannian manifold. For any smooth function  $u$  on  $S(M)$  we have the identity*

$$[H, X]u = X_\perp u_0 + X_\perp u_0 \tag{6}$$

where

$$u_0(x) = \frac{1}{2\pi} \int_{S_x} u(x, \xi) dS_x$$

is the average value.

*Proof.* It suffices to show that

$$[Id + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0.$$

Since  $X = \eta_+ + \eta_-$ , we need to compute  $[Id + iH, \eta_\pm]$ , so let us find  $[Id + iH, \eta_+]u$ , where  $u = \sum_k u_k$ . Recall that  $(Id + iH)u = u_0 + 2 \sum_{k \geq 1} u_k$ . We find:

$$\begin{aligned} (Id + iH)\eta_+ u &= \eta_+ u_{-1} + 2 \sum_{k \geq 0} \eta_+ u_k, \\ \eta_+(Id + iH)u &= \eta_+ u_0 + 2 \sum_{k \geq 1} \eta_+ u_k. \end{aligned}$$

Thus

$$[Id + iH, \eta_+]u = \eta_+ u_{-1} + \eta_+ u_0.$$

Similarly, we find

$$[Id + iH, \eta_-]u = -\eta_- u_0 - \eta_- u_1.$$

Therefore using that  $iX_\perp = \eta_+ - \eta_-$  we obtain

$$[Id + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0$$

as desired. □

We define

$$P_- = A_-^* H_- A_+, \quad P_+ = A_-^* H_+ A_+.$$

Separating the odd and even parts in (1.4) we get

$$H_+ \mathcal{H}u - \mathcal{H}H_- u = (\mathcal{H}_\perp u)_0, \quad H_- \mathcal{H}u - \mathcal{H}H_+ u = \mathcal{H}_\perp u_0$$

Take  $u = w_\psi$  with  $w \in C_\alpha^\infty(\partial_+ \Omega(M))$ . Then

$$2\pi \mathcal{H}H_+ w_\psi = -\mathcal{H}_\perp I^* w$$

using (1) we conclude

$$2\pi A_-^* H_- A_+ w = I \mathcal{H}_\perp I^* w \tag{7}$$

since  $w_\psi|_{\partial\Omega(M)} = A_+ w$ . Let  $h = I^* w$ , since  $I \mathcal{H}_\perp h = I \mathcal{H} h_* = -A_-^* h_*^0$ , one obtains

$$2\pi A_-^* H_- A_+ w = -A_-^* h_*^0. \tag{8}$$

## 5 The scattering relation and the Dirichlet-to-Neumann Map

Let  $(M, g)$  be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric  $g$  is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( \sqrt{\det g} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

where  $(g^{ij})$  is the inverse of the metric  $g$ . Let us consider the Dirichlet problem

$$\Delta_g u = 0 \text{ on } M, \quad u|_{\partial M} = f.$$

We define the DN map in this case by

$$\Lambda_g(f) = (\nu, \nabla u|_{\partial M})$$

The inverse problem is to recover  $g$  from  $\Lambda_g$ .

A similar obstruction to the one indicated in Section 1 holds for this problem. Namely

$$\Lambda_{\psi^*g} = \Lambda_g$$

where  $\psi$  is a  $C^\infty$  diffeomorphism of  $M$  which is the identity on the boundary.

In addition in the two dimensional case the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\beta g} = \frac{1}{\beta} \Delta_g$$

for any function  $\beta$ ,  $\beta \neq 0$ . Therefore we have that for  $n = 2$

$$\Lambda_{\beta(\psi^*g)} = \Lambda_g$$

for any non-zero  $\beta$  satisfying  $\beta|_{\partial M} = 1$ .

Therefore the best that one can do in two dimensions is to show that we can determine the conformal class of the metric  $g$  up to an isometry which is the identity on the boundary. That this is the case is a result proven in [23] for simple metrics and for general connected two dimensional Riemannian manifolds with boundary in [22].

More precisely we have:

**Theorem 5.1.** *Let  $(M, g)$  be a connected, compact Riemannian surface with boundary. Then  $(\Lambda_g, \partial M)$  determines uniquely the conformal class of  $(M, g)$ .*

As it was shown in [22] it is enough to measure the DN map in an open subset of the boundary.

The connection in two dimensions between the DN map and the scattering relation is given by

**Theorem 5.2.** *Let  $(M, g_i), i = 1, 2$ , be compact, simple two dimensional Riemannian manifolds with boundary. Assume that  $\alpha_{g_1} = \alpha_{g_2}$ . Then  $\Lambda_{g_1} = \Lambda_{g_2}$ .*

*End of Proof of Theorem 1.1*

The proof of Theorem 1.1 is reduced then to the proof of Theorem 5.2. In fact from Theorem 5.2 and Theorem 5.1 we get that we can determine the conformal class of the metric up to an isometry which is the identity on the boundary. Now by Theorem 1.2 we have that the conformal factor must be one proving that the metrics are isometric via a diffeomorphism which is the

identity at the boundary. In other words  $d_{g_1} = d_{g_2}$  implies that  $\alpha_{g_1} = \alpha_{g_2}$ . By Theorem 5.2  $\Lambda_{g_1} = \Lambda_{g_2}$ . By Theorem 5.1, there exists a diffeomorphism  $\psi : M \rightarrow M$ ,  $\psi|_{\partial M} = \text{Identity}$  and a function  $\beta \neq 0$ ,  $\beta|_{\partial M} = \text{identity}$  such that  $g_1 = \beta\psi^*g_2$ . By Mukhometov's theorem  $\beta = 1$  showing that  $g_1 = \psi^*g_2$  proving Theorem 1.1.

Before starting the proof of Theorem 5.2 we recall that Michel [25] has proven that for two dimensional manifolds Riemannian manifolds with strictly convex boundary one can determine from the boundary distance function, up to the natural obstruction, all the derivatives of the metric at the boundary. This result was generalized to any dimensions in [21].

The proof of Theorem 5.2 consists in showing that from the scattering relation we can determine the traces at the boundary of conjugate harmonic functions, which is equivalent information to knowing the DN map associated to the Laplace-Beltrami operator.

*Sketch of the proof of Theorem 5.2*

Let  $(h, h_*)$  be a pair of conjugate harmonic functions on  $M$ ,

$$\nabla h = \nabla_{\perp} h_*, \quad \nabla h_* = -\nabla_{\perp} h.$$

Notice, that  $\delta\nabla = \Delta$  is the Laplace-Beltrami operator and  $\delta\nabla_{\perp} = 0$ . Let  $I_0^*w = h$ . Since  $I_1\mathcal{H}_{\perp}h = I_1\mathcal{H}h_* = -A_-^*h_*^0$ , where  $h_*^0 = h_*|_{\partial M}$ , we obtain from the identity (6)

$$2\pi A_-^*H_+A_+w = -A_-^*h_*^0. \quad (9)$$

The following theorem gives the key to obtain the DN map from the scattering relation.

1

**Theorem 5.3.** *Let  $M$  be a 2D simple manifold. Let  $w \in C_{\alpha}^{\infty}(\partial_+\Omega(M))$  and let  $h_*$  be the harmonic extension of  $h_*^0$ . Then (6) holds iff  $h = I^*w$  and  $h_*$  are conjugate harmonic functions.*

*Proof.* One direction was done already. Assume (6) holds and let  $h = I^*w$ . By (5)

$$I\mathcal{H}_{\perp}h = I\mathcal{H}q$$

where  $q \in C^{\infty}(M)$  and  $q|_{\partial M} = h_*^0$ . Thus the ray transform of  $\nabla q + \nabla_{\perp}h$  vanishes, and it is known ([An]) that  $\nabla q + \nabla_{\perp}h = \nabla p$  with  $p|_{\partial M} = 0$ . Then  $h$  and  $h_* = q - p$  are conjugate harmonic functions and  $h_*|_{\partial M} = h_*^0$ .  $\square$

In summary we have the following procedure to obtain the DN map from the scattering relation. For a given smooth function  $h_*^0$  on  $\partial M$  we find a solution  $w \in C_\alpha^\infty(\partial_+ S(M))$  of the equation (1). Then the functions  $h^0 = 2\pi(A_+ w)_0$  (notice, that  $2\pi(A_+ w)_0 = I_0^* w|_{\partial M}$ ) and  $h_*^0$  are the traces of conjugate harmonic functions. It is easy to see that this gives the DN map.

## 6 Range and inversion of the geodesic X-ray transform

Let  $T(M)$  be the tangent bundle of  $M$ . We denote by  $\delta$  the divergence operator  $\delta : C^\infty(M, TM) \rightarrow C^\infty(M)$ . In local coordinates this is given by  $\delta u = g^{kj} \nabla_k u_j$  using Einstein's summation convention.

We define the operator  $\delta_\perp : C^\infty(M, T(M)) \rightarrow C^\infty(M)$  by

$$\delta_\perp u = -\delta u_\perp.$$

Then

$$\delta_\perp \nabla_\perp f = \delta \nabla f = \Delta f, \quad \delta_\perp \nabla f = -\delta \nabla_\perp f = 0.$$

We now give the characterization of the range of  $I_0$  and  $I_1$  in terms of the scattering relation only. We have that these are the projections of the operators  $P_-, P_+$  respectively. For the details see [32].

**Theorem 6.1.** *Let  $(M, g)$  be simple two dimensional compact Riemannian manifold with boundary. Then*

*i) The maps*

$$\begin{aligned} \delta_\perp I_1^* & : C_\alpha^\infty(\partial_+ S(M)) \rightarrow C^\infty(M), \\ \nabla_\perp I_0^* & : C_\alpha^\infty(\partial_+ S(M)) \rightarrow C_{sol}^\infty(M, T(M)) \end{aligned}$$

*are onto.*

*ii). A function  $u \in C^\infty(\partial_+ S(M))$  belong to  $\text{Range } I_0$  iff  $u = P_- w$ ,  $w \in C_\alpha^\infty(\partial_+ S(M))$ .*

*iii). A function  $u \in C^\infty(\partial_+ S(M))$  belong to  $\text{Range } I_1$  iff  $u = P_+ w$ ,  $w \in C_\alpha^\infty(\partial_+ S(M))$ .*

**Proposition 6.1.** *The operator  $W : C_0^\infty(M) \rightarrow C^\infty(M)$ , defined by*

$$Wf = (\mathcal{H}_\perp u^f)_0$$

*can be extended to a smoothing operator  $W : L^2(M) \rightarrow C^\infty(M)$ .*

We remark that in the case of constant Gaussian curvature  $W = 0$  and this does not depend on whether the metric has conjugate points so that the inversion formulas of Theorem 5.2 hold for all two dimensional manifolds with boundary with constant curvature.

The inversion formulas are (see [32])

**Theorem 6.2.** *Let  $(M, g)$  be a two-dimensional simple manifold. Then we have*

$$\begin{aligned} f + W^2 f &= \frac{1}{2\pi} \delta_{\perp} I_1^* w, & w &= \frac{1}{2} \alpha^* H(I_0 f)^- |_{\partial_+ S(M)}, & f &\in L^2(M), \\ h + (W^*)^2 h &= \frac{1}{2\pi} I_0^* w, & w &= \frac{1}{2} \alpha^* H(I_1 \mathcal{H}_{\perp} h)^+ |_{\partial_+ S(M)}, & h &\in H_0^1(M), \end{aligned}$$

where  $W, W^* : L^2(M) \rightarrow C^{\infty}(M)$ . In the case of a manifold of constant curvature  $W = 0$ ,  $W^* = 0$ .

## 7 Remarks

The Hilbert transform for 2-dimensional Riemannian manifolds is the map that relates the restrictions on the boundary of conjugate harmonic functions. In this sense the Hilbert transform, up to a constant, is just the Dirichlet-to-Neumann (DN) map. In Section 4 we fixed a point  $x$  and started with the microlocal Hilbert transform on the circle  $S_x$  in the tangent space with Euclidean metric and we ended with the global Hilbert transform (the DN map).

Comparison of boundary rigidity and EIT

$d_g/\alpha_g$	$\Lambda_g$
function of $2n - 2$ variables	kernel is a function of $2n - 2$ variables
$d_{\psi^*g} = d_g$	$\Lambda_{\psi^*g} = \Lambda_g$
singularities determine $\partial^{\alpha} g _{\partial M}$	singularities determine $\partial^{\alpha} g _{\partial M}$
$d_g(x, y) = \inf_{\substack{\gamma(a)=x \\ \gamma(b)=y}} \int_a^b \sqrt{g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t)} dt$	$Q_g(f) = \inf_{u _{\partial M}=f} \int_M \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$
graph of $\alpha_g$ is a Lagrangian manifold (finite dimension)	graph of $\Lambda_g$ is a Lagrangian manifold (infinite dimension)
obtained from solutions of Hamilton ODE	obtained from solution of PDEs

The scattering relation and the boundary distance function are determined by the singularities of the DN map associated to the wave equa-

tion for the Laplace-Beltrami operator, the so-called hyperbolic (or dynamic) Dirichlet-to-Neumann map [39]. We have found, in two dimensions, a connection between the scattering relation and the *elliptic* Dirichlet-to-Neumann map which led to a solution of the boundary rigidity problem in two dimensions. Is there a similar connection in higher dimensions?

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