Recovering discontinuous conductivity from internal current : case of the ultrasonically-induced Lorentz force electrical impedance tomography

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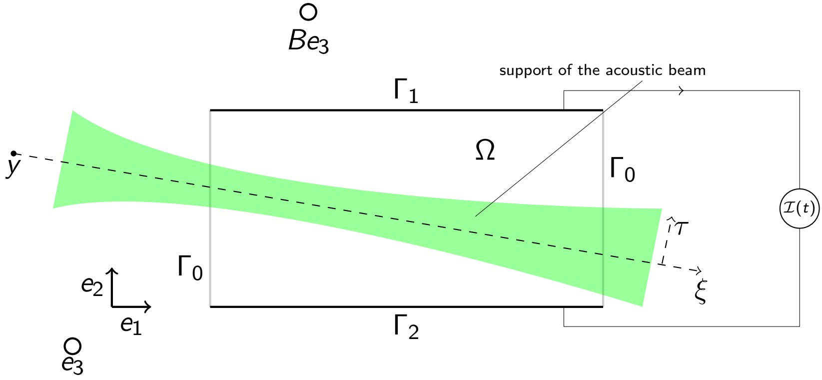
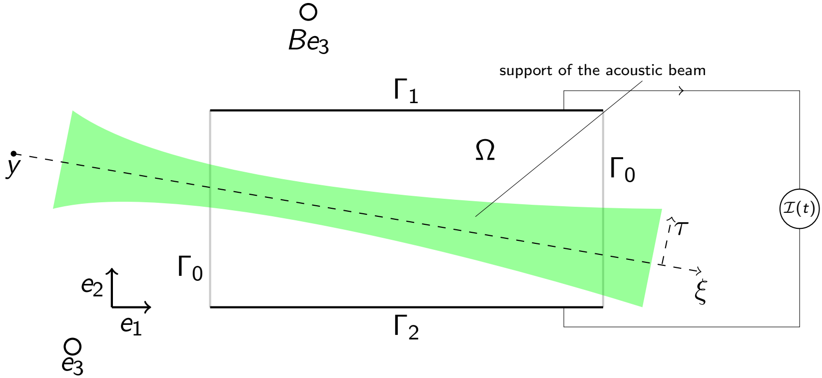
projet ERC : MULTIMOD

1. How to create currents with an acoustic beam and a constant magnetic field ?
   - The ultrasonically induced Lorenz force tomography
   - Ionic description of the conductivity in aqueous tissues
   - Boundary measurements
2. From boundary measurements to meaningful internal data
   - Introduction of a virtual potential
   - Deconvolution
   - Geometric integral transform or asymptotic formula
3. Recovering the conductivity from an internal current
   - By optimization
   - By solving a transport equation



Assumptions

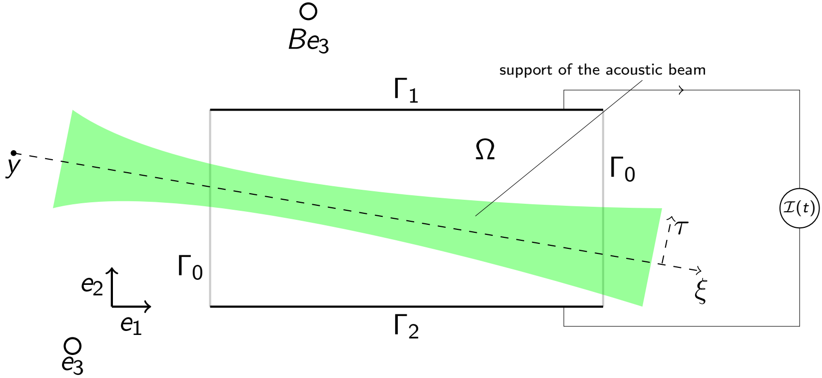
Ω mechanically homogeneous and is a conductive medium. Γ_1 and Γ_2 are perfect conductors. Γ_0 is a perfect isolator. B is constant.



Velocity field

For any $x \in \Omega$, written $x = y + z\xi + r$ with $z > 0$, $r \in \xi^\perp$,

$$v_{y,\xi}(y + z\xi + r, t) = A(z, |r|)w(z - ct)\xi$$



As Ω is electrically neutral, can we explain the origin of the current measured at the electrodes ?

Assume that Ω is an electrolyte medium (saline gel, living tissues,...) the conductivity phenomenon is due to the presence of ions. Assume that we have N types of ions of charge q_i and volume density $n_i(x)$, $i \in \{1, \dots, N\}$. We have, for any $x \in \Omega$

Neutrality

$$\sum_i q_i n_i(x) = 0$$

Kolhrausch's law

$$\sigma(x) = e^+ \sum_i \mu_i q_i n_i(x)$$

with $\mu_i \in \mathbb{R}$, satisfying $\mu_i q_i > 0$ is called the ionic mobility and e^+ is the elementary charge.

We can understand now understand the source of current as the deviation of the ions by the magnetic field B.

Consider an ion i at position x at time t. The acoustic beam imposes to it a velocity in the direction ξ : $v(x,t)\xi$. The Lorentz force applied to i is

$$F_i = q_i v\xi \times Be_3$$

and the ion get almost immediately an additional drift speed

$$v_{d,i} = \frac{\mu_i}{q_i} F_i = B\mu_i v\tau$$

where $\tau = \xi \times e_3$. At first order in the displacement length, its total velocity is

$$v_i = v\xi + B\mu_i v\tau.$$

Defining the current as the total amount of charges displacement,

$$j_S = \sum_i n_i q_i v_i = \left(\sum_i n_i q_i\right) v\xi + B\left(\sum_i n_i \mu_i q_i\right) v\tau = \frac{B}{e^+}\sigma v\tau.$$

The interaction between the velocity field $v(x,t)\xi$ and the magnetic field Be_3 create a source of current

$$j_S(x,t) = \frac{B}{e^+}\sigma(x)v(x,t)\tau$$

Our measure is the indirect effect of j_S on the boundary. Assume that the electromagnetic propagation is much faster than the acoustic propagation, we adopt the electrostatic approximation.

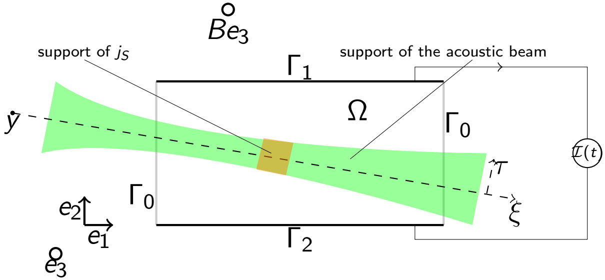
$$j = j_S + \sigma\nabla u$$

satisfying

$$\nabla \cdot j = 0$$

then the potential satisfies at a fixed time t,

$$-\nabla \cdot (\sigma\nabla u) = \nabla \cdot j_S \quad \text{in } \Omega$$



$$u : \begin{cases} -\nabla \cdot (\sigma \nabla u) = \nabla \cdot j_S & \text{in } \Omega \\ u = 0 & \text{on } \partial\Gamma 1 \cup \Gamma_2 \\ \partial_\nu u = 0 & \text{on } \Gamma_0 \end{cases}$$

The intensity that we measure is

$$I = \int_{\Gamma_2} \sigma \partial_\nu u$$

In order to understand the measurements, we multiply the potential equation by a well chosen test function U called virtual potential defined by

$$\begin{cases} -\nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega \\ U = 0 & \text{on } \Gamma 1 \\ U = 1 & \text{on } \Gamma 2 \\ \partial_\nu U = 0 & \text{on } \Gamma_0 \end{cases}$$

and through integration by part it comes

$$I = \int_\Omega j_S \cdot \nabla U = \frac{B}{e^+} \int_\Omega v(x,t) \sigma(x) \nabla U(x) dx \cdot \tau$$

and we define the measurments function as

$$M_{y,\xi}(z) = \int_\Omega v_{y,\xi} \left(x, \frac{z}{c} \right) \sigma(x) \nabla U(x) dx \cdot \tau_\xi$$

The inverse problem posed by this hybrid method is

Inverse problem

Find $\sigma : \Omega \to \mathbb{R}$ from the knowledge of

$$M_{y,\xi} : z \to \int_{\Omega} v_{y,\xi}\left(x, \frac{z}{c}\right) \sigma(x) \nabla U(x) dx \cdot \tau_{\xi}$$

known for any $y \in Y \subset \mathbb{R}^d$ and $\xi \in \Theta \subset S^{d-1}$

In general, Y is supposed to be a bounded smooth surface of $\mathbb{R}^d$.

Idea

If Y and Θ are well chosen, we show that the virtual current $J(x) = (\sigma \nabla U)(x)$ can be recovered.

Step 1 : Deconvolution

As $v_{y,\xi}\left(y+z'\xi+r,\frac{z}{c}\right)=w(z'-z)A(z',|r|)$ we rewrite the measurments $M_{y,\xi}$ as

$$M_{y,\xi}(z)=(w*\Phi_{y,\xi})(z)$$

where

$$\Phi_{y,\xi}(z)=\int_{\xi^\perp}(\sigma\nabla U)(y+z\xi+r)A(z,|r|)dr\cdot\tau_\xi$$

To recover $\Phi_{y,\xi}$ with stability, we need short pulses and/or changes of the frequency. Tu recover the largest spectral band in the Fourrier domain.

Step 2 : Getting the current

Once we know

$$\Phi_{y,\xi}(z) = \int_{\xi^\perp} (\sigma \nabla U)(y + z\xi + r) A(z, |r|) dr \cdot \tau_\xi$$

we can notice that it looks like a weighted Radon transform of the current density. If we assume that the support of A is thin,

$$\Phi_{y,\xi}(z) = (\sigma \nabla U)(y + z\xi) \int_{\xi^\perp} A(z, |r|) dr \cdot \tau_\xi + \mathcal{O}(R)$$

where R is such that $\text{supp}(\rho \mapsto A(z, \rho)) \subset [0, R]$ and with a remainder depending on $|\sigma \nabla U|_{TV(\Omega)}$. Finally, choosing $x \in \Omega$ and consider $\Phi_{y,\xi}(z)$ for any (y, ξ, z) such that $x = y + z\xi$ we reconstruct

$$J(x) = (\sigma \nabla U)(x)$$

Now the problem is the following,

Recover σ from $J = \sigma \nabla U$ where U is defined as

$$U = F[\sigma] = \begin{cases} -\nabla \cdot (\sigma \nabla U) = 0 & \text{in } \Omega \\ U = 0 & \text{on } \Gamma 1 \\ U = 1 & \text{on } \Gamma 2 \\ \partial_\nu U = 0 & \text{on } \Gamma_0 \end{cases}$$

Classical approach is to minimize

$$K_\varepsilon[\sigma] = \frac{1}{2} \int_\Omega |\sigma \nabla F[\sigma] - J|^2 + \varepsilon |\sigma|_{TV(\Omega)}$$

This works but the convexity is not good (numerically).

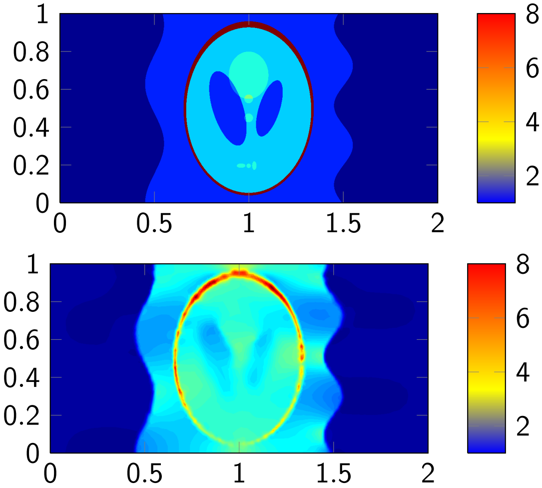


Figure: Conductivity map σ to be reconstructed and the reconstroction by optimisation.

Orthogonal field transport equation

If we know $\sigma\nabla U$, we know the direction of ∇U. From this we can try to reconstruct the potential U. Let us construct a vectorial field F such that

$$\nabla U \cdot F = 0 \quad \text{in } \Omega$$

and $U|_{\Gamma_1} = 0$, $U|_{\Gamma_2} = 1$ and if the variations of σ are supposed far from Γ_0, we can look for U in $H^1(\Omega)$ as a solution of

$$\begin{cases} F \cdot \nabla U = 0 & \text{in } \Omega \\ U = x_2 & \text{on } \partial\Omega \end{cases}$$

This idea is good only if the previous problem admits a unique solution !

The transport problem

$$\begin{cases} F \cdot \nabla U = 0 & \text{in } \Omega \\ U = x_2 & \text{on } \partial\Omega \end{cases}$$

is highly related to the corresponding characteristic flow problem

$$\begin{cases} \partial_t X(x,t) = F(X(x,t)) & \text{on } [0,T[\\ X(x,0) = x \in \Omega & \end{cases}$$

because $t \mapsto U(X(x,t))$ would be a constant function. We would need F to be local Lipschitz in Ω...

Problem

F is not even continuous !

About Cauchy problem with non smooth field

Theorem [DiPerna-Lions 89]

Consider $u \in L^1(\Omega)$ satisfying

$$\begin{cases} F \cdot \nabla u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $F \in L^1(\Omega) \cap W^{1,1}_{loc}(\Omega)^d$, $\nabla \cdot F \in L^\infty(\Omega)$, then

$$u = 0.$$

Controlling the divergence is necessary to control the measure transport by the flow. We have

$$e^{-ct}\lambda \leq \lambda \circ X(t) \leq \lambda e^{ct}$$

where $c = \|\nabla \cdot F\|_{L^\infty(\Omega)}$ and λ is the Lebesgue measure. Basically, this prevents two different characteristic lines from touching each other. Then Lions in 96 extended it to "piecewise" $W^{1,1}$ regularity.

And with BV regularity ?

Theorem [Ambrosio 03]

Assume that $F \in L^\infty(\Omega) \cap BV_{loc}(\Omega)$, $\nabla \cdot F \in L^\infty_{loc}(\Omega)$, then there exists a unique lagrangian flow X satisfying

$$X(x,t) = x + \int_0^t F(X(x,u))du.$$

That would assure the uniqueness for our transport equation. But in our case if we compute formally $\nabla \cdot F = \nabla \cdot (\sigma \nabla U \times e_3) = \nabla \sigma \times \nabla U \cdot e_3 +$ something. No chance to fit in $L^\infty(\Omega)$ even locally. We shall try another approach.

We remarked that we need only existence of a flow and we do not really care about uniqueness. To fixe the ideas,

existence of outgoing flow $\Rightarrow$ uniqueness for the transport

Theorem [Bressan-Shen 98]

Assume that $F(x) = g(\tau(x), x)$ where
$\tau : \mathbb{R}^d \to \mathbb{R}$ is C^1, $t \mapsto g(t,x)$ is measurable
$x \mapsto g(t,x)$ is Lipschitz.
If there exist a compact set K such that $f(x) \in K$ and
$\nabla\tau(x) \cdot z > 0$ for all $x \in \Omega$, $z \in K$ Then the Cauchy problem

$$\begin{cases} \partial_t X(x,t) = F(X(x,t)) \text{ on } [0, T[\\ X(x,0) = x \in \Omega \end{cases}$$

has at least solution.

Problem : F cannot be tangent to its own discontinuities. This is called by Bressan the "transversality condition".

Dead end ?

Our flow cannot be Lagranian so neither fits with the DiPerna-Lions theory nor the Ambrosio's one. The flow can be tangent to the discontinuities so it does not fit with the Bressan-Shen Cauchy problem.

We can try our own (local) existence of a characteristic flow which may fit our problem.

For any surface $S \in \Omega$ of class C^2 cutting Ω in connected Lipschitz domains Ω_i, we say that $f \in C_S^{k,\alpha}(\overline{\Omega})$ if $f|_{\Omega_i} \in C^{k,\alpha}(\overline{\Omega}_i)$

Theorem : Local existence for characteristic flow

Consider a smooth surface $S \subset \Omega$ and $F \in C_S^{k,\alpha}(\overline{\Omega})^2$. Assume that the jump of F on S can be written

$$F^+ = f\tau + gh^+\nu$$
$$F^- = f\tau + gh^-\nu$$

where ν is the normal to S and τ the tangent vector and with f, g, h^+ and h^- are in $C^{0,\alpha}(S)$, h^+ and h^- are positive and g locally signed. Then for any $x \in \Omega$, there exists $T > 0$ and $X \in C^1([0,T],\Omega)$ such that $t \mapsto F(X(t))$ is measurable and

$$X(t) = x + \int_0^t F(X(s))ds \quad \forall t \in [0,T[.$$

Enough difficulties ! To assure that the characteristics reach the boundary, we add the hypothesis

$$F \cdot e_1 \geq c > 0$$

Theorem : Existence of outgoing characteristics

If F satisfies the previous conditions, for any $x \in \Omega$ there exists $T \in]0, diam(\Omega)/c[$ and $X \in \mathcal{C}^0([0, T[, \Omega)$ such that $t \mapsto F(X(t))$ is measurable and

$$X(t) = x + \int_0^t F(X(s))ds \quad \forall t \in [0, T[$$

and

$$\lim_{t \to T} X(t) \in \partial\Omega.$$

We have a uniqueness result,

Corollary

If F satisfies the previous conditions, and $u \in C^0(\overline{\Omega}) \cap C_S^{0,\alpha}(\overline{\Omega})$ satisfies

$$\begin{cases} F \cdot \nabla u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $u = 0$ in Ω.

If the current is such that F satisfies all the previous conditions, the virtual potential U can be found solving

$$\begin{cases} F \cdot \nabla U = 0 & \text{in } \Omega \\ U = x_2 & \text{on } \partial\Omega, \end{cases}$$

To solve this we introduce the regularized problem

$$\begin{cases} -\nabla \cdot (\varepsilon(I + FF^T)\nabla U_\varepsilon) = 0 & \text{in } \Omega \\ U_\varepsilon = x_2 & \text{on } \partial\Omega, \end{cases}$$

and prove

Proposition

The sequence $(U_\varepsilon - U)_{\varepsilon>0}$ converges strongly to zero in $H_0^1(\Omega)$.

Sketch of proof :

- $\nabla(U_\varepsilon - U)$ is bounded in $L^2(\Omega)$
- up to an extraction $(U_\varepsilon - U)$ converges in $H_0^1(\Omega)$ for the *weak* $- *$ topology.
- The limit U^* satisfies

$$\begin{cases} F \cdot \nabla U^* = 0 & \text{in } \Omega \\ U^* = 0 & \text{on } \partial\Omega, \end{cases}$$

so using the previous work, $U^* = 0$.

- We prove that the convergence is strong and we do not need extraction.

Corollary

The sequence $\dfrac{1}{\sigma_\varepsilon} := \frac{J \cdot \nabla U_\varepsilon}{|J|^2}$ converges to $\dfrac{1}{\sigma}$ strongly in $L^2(\Omega)$.

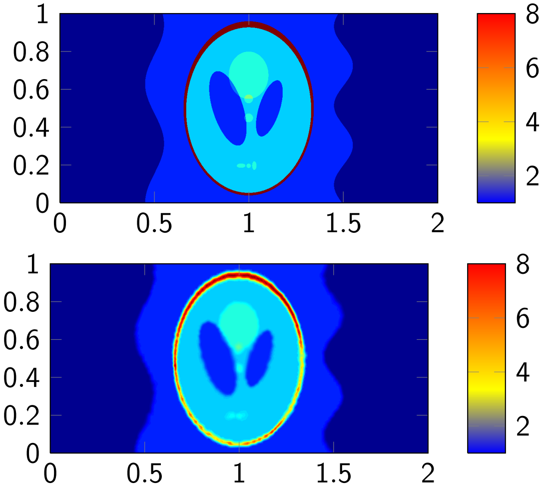


Figure: Conductivity map σ to be reconstructed and the reconstruction through transport equation.