

Higher representation theory in algebra and geometry: Lecture I

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January 30, 2014

References

For this lecture, useful references include:

- Beilinson, Lusztig and MacPherson, *A geometric setting for the quantum deformation of GL_n* (covers the appearance of \mathfrak{sl}_2 by counting points in Grassmannians)
- Chuang and Rouquier, *Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification* (introduces notion of weak \mathfrak{sl}_2 categorification and covers the appearance of \mathfrak{sl}_2 in cohomology of Grassmannian (briefly))
- Lauda, *A categorification of quantum $\mathfrak{sl}(2)$ and Categorified quantum $\mathfrak{sl}(2)$ and equivariant cohomology of iterated flag varieties* (gives a more detailed account of connection to the cohomology of Grassmannians)

The slides for the talk are on my webpage at:

<http://people.virginia.edu/~btw4e/lecture-1.pdf>

My justification

So, the first question I should address is this: why should you show up for this course? Why is what I'm about to tell you worth 20 hours of your time, when you could be at home petting your dog?

I'll get into the specific content of in a moment, but let me put in a pitch for the whole field of "categorification."

Definition

cat·e·gor·i·fic·a·tion *noun*

the philosophy that any number worth its salt is the size of a set or dimension of a vector space, and any interesting set is the simple objects in category (even if it doesn't know it yet).

My justification

I think this idea tends to sound like a scam when people first hear it. They tend to say things like “wait, but that isn’t going to be unique...”

I’m contractually obligated to reply “categorification is an art, not a functor.”

Sometimes, things in life aren’t unique, but they can still be useful.

Example

The rational function

$$\frac{x_1^{100}x_2^{73} - x_2^{100}x_1^{73} + x_2^{100}x_3^{73} - x_3^{100}x_2^{73} + x_3^{100}x_1^{73} - x_1^{100}x_3^{73}}{x_1^2x_2 - x_2^2x_1 + x_2^2x_3 - x_3^2x_2 + x_3^2x_1 - x_1^2x_3}$$

is actually a polynomial with non-negative integral coefficients; Weyl tells us that these are the weight multiplicities of a simple representation of GL_3 .

Might seem like a forced example, but who would have thought to look at Schur functions if they hadn’t known about GL_n ?

My justification

Another example is the replacement of Euler characteristic by homology.

- homology is a stronger invariant of topological spaces than Euler characteristic, for sure.
- much more important, it's functorial! You can talk about maps between sets or vector spaces, not between numbers.

While algebraic topology certainly does a good job of telling topological spaces apart, it would be pretty reductive to suggest that was its main goal. It also gives us a lens for thinking about the structure of topology which couldn't even be phrased just taking about numbers.

My justification

So these are the themes I want to emphasize throughout.

Categorifications:

- allow us to show properties of weaker structures that might be hard to see otherwise. Positivity and integrality results are especially important, but there are other examples.
- suggest facets of already understood structures that are interesting on their own, but we might not have noticed. In particular, I maintain that the theory of quantum groups makes absolutely no sense unless you categorify it.
- allow us to ask questions about previously known invariants and structures that simply made no sense before. Think Euler characteristic vs. homology.

My plan

Ok, after that long prolegomenon, let me give a short plan for the course. It neatly divides into two halves (at least in theory): one on the categorification of structures related to \mathfrak{sl}_2 , and a second generalizing these notions to other Lie algebras.

- 1 in the first half, we'll cover the definition of a categorical action of \mathfrak{sl}_2 , its connection to the cohomology of Grassmannians, how to categorify simple representations and tensor products and connections to Khovanov homology and the Temperley-Lieb category.
- 2 in the second half, we'll extend this idea to other Lie algebras. This will require defining KLR algebras. As time permits, we'll discuss different applications of this formalism, in knot theory, the theory of symmetric groups, Hecke algebras and Cherednik algebras.

About \mathfrak{sl}_2

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$. This is the Lie algebra of 2×2 trace 0 matrices with the usual commutator.

If we let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then this algebra has a presentation of the form

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H$$

By definition, the universal enveloping algebra $U(\mathfrak{sl}_2)$ is the *associative* algebra generated by the symbols E, F, H subject to the relations above (where $[-, -]$ means commutator).

About \mathfrak{sl}_2

This algebra is fundamental for a lot of reasons, but let me emphasize a slightly less well-known one: it appears naturally from the structure of the category of finite sets.

Fix a finite set S , and consider the space V of functions $2^S \rightarrow \mathbb{C}$ from the power set of S . This is a \mathbb{C} -vector space of dimension $2^{|S|}$.

Consider the operators E and F on V defined by

$$E \cdot g(T) = \sum_{s \in S \setminus T} g(T \cup \{s\}) \quad F \cdot g(T) = \sum_{t \in T} g(T \setminus \{t\})$$

A ridiculously easy, but still rather nice, theorem is:

Theorem

The operators E and F induce an action of \mathfrak{sl}_2 on V . A function supported on sets of size k has weight (H -eigenvalue) $|S| - 2k$.

About \mathfrak{sl}_2

This is one of these theorems that gets less and less impressive as you think about it.

Proof. First, note that if $S = S_1 \cup S_2$, then $V_S \cong V_{S_1} \otimes V_{S_2}$, and that the endomorphisms E and F satisfy:

$$E_S = E_1 \otimes 1 + 1 \otimes E_2 \quad F_S = F_1 \otimes 1 + 1 \otimes F_2.$$

Thus, if we have an \mathfrak{sl}_2 -action on V_{S_1} and V_{S_2} , then the action of V_S is just the tensor product of these. Thus, we reduce to the case of a singleton: $S = \{s\}$.

In this case, V has a basis given by the characteristic functions v_0 of \emptyset and v_1 of $\{s\}$. In this case,

$$E v_0 = 0 \quad E v_1 = v_0 \quad F v_1 = 0 \quad F v_0 = v_1.$$

This is how \mathfrak{sl}_2 acts on \mathbb{C}^2 , if we identify $v_0 \mapsto (1, 0)$ and $v_1 \mapsto (0, 1)$.

\mathfrak{sl}_2 and Grassmannians

Ok, so this was just a fancy way of writing $(\mathbb{C}^2)^{\otimes |S|}$. But it's not completely bankrupt; in fact, this is the kernel of the reason that \mathfrak{sl}_2 has a categorification.

So, how can we upgrade this? Well, secretly, the power set 2^S is “the points of the Grassmannians in S over a field with 1 element.” Maybe that's an insane statement to make, but certainly any interesting game involving inclusions of finite sets can be played using inclusions of vector spaces, especially of a finite field.

Let S be a vector space over a finite field \mathbb{F}_p , let X_S denote the union of the Grassmannians of subspaces in S (just as sets), and let V_S denote the vector space of functions $X_S \rightarrow \mathbb{C}$.

\mathfrak{sl}_2 and Grassmannians

We can define some operators E and F on V_S much like before. For notational purposes, we set $d = \dim(S)$ and $k = \dim(W)$.

$$E \cdot g(W) = p^{-1/2(k-1)} \sum_{\substack{W' \supset W \\ \dim(W'/W)=1}} g(W')$$

$$F \cdot g(W) = p^{-1/2(d-k-1)} \sum_{\substack{W' \subset W \\ \dim(W/W')=1}} g(W')$$

Ok, some of you know what the next theorem is. Some of you think you do, but you're wrong. These *don't* define an action of \mathfrak{sl}_2 in a literal sense, though perhaps they do in a moral one.

\mathfrak{sl}_2 and Grassmannians

In order to see what the problem is, let's compute the commutator $[E, F]$:

$$\begin{aligned} [E, F] \cdot g(W) &= EF \cdot g(W) - FE \cdot g(W) \\ &= p^{-1/2(d-1)} \sum_{\substack{W'' \subset W' \subset W \\ \dim(W'/W) = \dim(W'/W'') = 1}} g(W'') - p^{-1/2(d-1)} \sum_{\substack{W'' \supset W' \subset W \\ \dim(W/W') = \dim(W''/W') = 1}} g(W'') \end{aligned}$$

If $W'' \neq W$, it appears exactly once in both sums, and thus its contributions cancel. On the other hand, W itself appears $\#\mathbb{P}(S/W)$ many times in the first term, and $\#\mathbb{P}(W)$ many times in the second. Thus, we have

$$[E, F] \cdot g(W) = p^{-1/2(d-1)} (\#\mathbb{P}(S/W) - \#\mathbb{P}(W))g(W)$$

This scalar is given by

$$\begin{aligned} p^{1/2d-k-1} + p^{1/2d-k-2} + \dots + p^{-1/2(d-1)} - p^{k-1/2d-1} - p^{k-1/2d-2} - \dots - p^{-1/2(d-1)} \\ = \frac{p^{1/2d-k} - p^{-1/2d+k}}{p^{1/2} - p^{-1/2}} \end{aligned}$$

\mathfrak{sl}_2 and Grassmannians

This is a *lot* like a \mathfrak{sl}_2 representation. After all, $[E, F]$ acts by a scalar on the functions concentrated in a single dimension (which thus are like weight spaces).

However, for an \mathfrak{sl}_2 representation, we would have expected that the scalar would be $d - 2k$, not $[d - 2k]_{p^{1/2}} = \frac{p^{1/2d-k} - p^{-1/2d+k}}{p^{1/2} - p^{-1/2}}$.

As a function of a formal variable q , we call $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ the **quantization** or **quantum number** of n ; note that $\lim_{q \rightarrow 1} [n]_q = n$.

Theorem (Beilinson-Lusztig-Macpherson)

Actually E and F induce an action of the **quantized universal enveloping algebra** $U_q(\mathfrak{sl}_2)$, a flat deformation of $U(\mathfrak{sl}_2)$ over $\mathbb{C}[q, q^{-1}]$, specialized at $q = p^{1/2}$.

However, I refuse to define $U_q(\mathfrak{sl}_2)$ before defining its categorification, so don't worry too much about this difference.

Convolution

The operations E and F can be thought of as having a geometric identity themselves: for any finite set X , the set of functions $\mathbb{C}[X \times X]$ has an algebra structure under convolution:

$$f \star g(x_1, x_2) = \sum_{x' \in X} f(x_1, x')g(x', x_2).$$

It's easily shown that this is associative, and acts naturally on $\mathbb{C}[X]$ by

$$f \star h(x) = \sum_{x' \in X} f(x, x')h(x').$$

Thus, the operations E and F are convolution with the functions

$$e(W, W') = \begin{cases} p^{-1/2 \dim W} & W \subset W', \dim(W'/W) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(W, W') = \begin{cases} p^{-1/2 \dim(S/W)} & W \supset W', \dim(W/W') = 1 \\ 0 & \text{otherwise} \end{cases}$$

Back to categorification

I think you could rightly say “OK, that’s all lovely, but what does it have to do with the price of tea in China?” (By which I mean categorification..)

Again, I suspect some of you know what’s coming. As a hint, the set I’ve called X_S isn’t really just a set. It’s the \mathbb{F}_p points of an algebraic variety. So, functions on it are avatars of sheaves, under the function-sheaf correspondence.

So, in order to categorify, I should replace every function by a sheaf.

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So, in order to categorify, I should replace every function by a sheaf. Or not. This is the first class; maybe it’s a little early for the perverse sheaves.

Cohomology and bimodules

Definition

There is a decent replacement for them, though: the categories of modules over cohomology rings. Let S be a complex vector space now, and let $\mathbf{Gr}(k, S)$ be the Grassmannian. We'll want to consider the categories of modules over the rings $H_k := H^(\mathbf{Gr}(k, S); \mathbb{C})$ as a replacement for the vector space V_S .*

The maps E and F between these spaces should turn into functors, connected to the subspace

$$\mathbf{Gr}(k \subset k + 1, S) = \{(W, W') \in \mathbf{Gr}(k, S) \times \mathbf{Gr}(k + 1, S) \mid W \subset W'\}.$$

Cohomology and bimodules

We have a diagram of projections:

$$\begin{array}{ccc} & \text{Gr}(k \subset k+1, S) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \text{Gr}(k, S) & & \text{Gr}(k+1, S) \end{array}$$

Definition

Let ${}_k H_{k+1} := H^*(\text{Gr}(k \subset k+1, S))$ as a left module over H_k via π_1^* and a right module over H_{k+1} via π_2^* .

Let ${}_{k+1} H_k := H^*(\text{Gr}(k \subset k+1, S))$ with these actions reversed.

Consider the functors

$$\mathcal{E}^k(M) := {}_{k-1} H_k \otimes_{H_k} M \quad \mathcal{F}^k(M) := {}_{k+1} H_k \otimes_{H_k} M.$$

Cohomology and bimodules

Principle

The functors \mathcal{E} and \mathcal{F} categorify the operators E and F .

What the heck does this mean? I hope that intuitively, this seems reasonable: before, I had the characteristic function of a subvariety, and now I've replaced that with its cohomology, a categorically richer object.

Theorem (Beilinson-Lusztig-Macpherson?, Chuang-Rouquier)

$${}_k H_{k+1} \otimes_{H_{k+1}} {}_{k+1} H_k \cong ({}_k H_{k-1} \otimes_{H_{k-1}} {}_{k-1} H_k) \oplus H_k^{\oplus d-2k} \quad (k \leq d/2)$$

$$({}_k H_{k+1} \otimes_{H_{k+1}} {}_{k+1} H_k) \oplus H_k^{\oplus 2k-d} \cong {}_k H_{k-1} \otimes_{H_{k-1}} {}_{k-1} H_k \quad (k \geq d/2)$$

That is, a module over H_k behaves like a vector of weight $d - 2k$ under the action of \mathcal{E} and \mathcal{F} .

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Theorem (Beilinson-Lusztig-Macpherson?, Chuang-Rouquier)

$$\mathcal{E}^{k+1} \mathcal{F}^k \cong \mathcal{F}^{k-1} \mathcal{E}^k \oplus \text{id}^{\oplus d-2k} \quad (k \leq d/2)$$

$$\mathcal{E}^{k+1} \mathcal{F}^k \oplus \text{id}^{\oplus 2k-d} \cong \mathcal{F}^{k-1} \mathcal{E}^k \quad (k \geq d/2)$$

That is, a module over H_k behaves like a vector of weight $d - 2k$ under the action of \mathcal{E} and \mathcal{F} .

A first categorical representation

Idea of proof

It's a little difficult to give a proper proof without unpleasant formulae or getting a bit more sheafy, but the underlying idea is simple.

First note that:

$${}_k H_{k+1} \otimes_{H_{k+1}} {}_{k+1} H_k \cong H^*(\mathrm{Gr}(k \subset k+1 \supset k, S))$$

$${}_k H_{k-1} \otimes_{H_{k-1}} {}_{k-1} H_k \cong H^*(\mathrm{Gr}(k \supset k-1 \subset k, S))$$

There's a birational map between these varieties:

$$\{W'' \supset W' \subset W\} \mapsto \{W'' \subset W'' + W \supset W\} \quad \text{when } W'' \neq W$$

The “difference” between the two cohomologies is the same as the “difference” between the cohomologies of the bad sets for this map as modules over $H^*(\mathrm{Gr}(k, S))$.

A first categorical representation

One bad set is $H^*(\text{Gr}(k \subset k+1, S))$ and the other $H^*(\text{Gr}(k \supset k-1, S))$. The maps

$$H^*(\text{Gr}(k \subset k+1, S)) \rightarrow H^*(\text{Gr}(k, S)) \quad H^*(\text{Gr}(k \supset k-1, S)) \rightarrow H^*(\text{Gr}(k, S))$$

are fiber bundles with fiber \mathbb{P}^{d-k-1} in the first case and \mathbb{P}^{k-1} in the other. Thus, the cohomologies are free H_k -modules with ranks k and $d-k$.

Thus, the “difference” is

$${}_k H_{k+1} \otimes_{H_{k+1}} {}_{k+1} H_k \text{ “} - \text{” } {}_k H_{k-1} \otimes_{H_{k-1}} {}_{k-1} H_k \text{ “} = \text{” } (d-2k) \cdot H_k$$

Gradings

Thus far, everything I've done has been with ungraded modules, but the algebras H_k are graded and we can make this whole construction homogeneous.

Definition

As a graded bimodules, let

$${}_{k-1}H_k := H^*(\mathrm{Gr}(k \subset k+1, S))(k-1)$$

$${}_{k+1}H_k := H^*(\mathrm{Gr}(k \subset k+1, S))(d-k-1)$$

where $M(a)$ or $q^a M$ is M with its grading decreased by a .

Now, we should prove a graded version of the categorification theorem.

Gradings

Theorem (Beilinson-Lusztig-Macpherson?, Lauda)

$$\mathcal{E}^{k+1}\mathcal{F}^k \cong \mathcal{F}^{k-1}\mathcal{E}^k \oplus \text{id}(d-2k) \oplus \text{id}(d-2k-2) \oplus \cdots \oplus \text{id}(2k-d) \quad (k \leq d/2)$$

$$\mathcal{E}^{k+1}\mathcal{F}^k \oplus \text{id}(2k-d) \oplus \text{id}(d-2k-2) \oplus \cdots \oplus \text{id}(d-2k) \cong \mathcal{F}^{k-1}\mathcal{E}^k \quad (k \geq d/2)$$

That is, paying attention to the grading, we see that an H_k -module behaves like a vector of weight $d - 2k$ for the quantum group $U_q(\mathfrak{sl}_2)$. Setting $q = 1$ corresponds to ignoring the grading.

This proof is almost the same as the ungraded proof; one just pays attention to the grading on the cohomology of the bad loci. The term

$$[d - 2k]_q = q^{d-2k} + q^{d-2k-2} + \cdots + q^{2k-d}$$

thus comes from the difference of the cohomology of \mathbb{P}^{d-k-1} and \mathbb{P}^{k-1} (suitably shifted).

Gradings

Theorem (Beilinson-Lusztig-Macpherson?, Lauda)

$$\mathcal{E}^{k+1} \mathcal{F}^k \cong \mathcal{F}^{k-1} \mathcal{E}^k \oplus \mathrm{id}^{\oplus [d-2k]_q} \quad (k \leq d/2)$$

$$\mathcal{E}^{k+1} \mathcal{F}^k \oplus \mathrm{id}^{\oplus [2k-d]_q} \cong \mathcal{F}^{k-1} \mathcal{E}^k \quad (k \geq d/2)$$

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thus comes from the difference of the cohomology of \mathbb{P}^{d-k-1} and \mathbb{P}^{k-1} (suitably shifted).

Upgrade to sheaves

Don't worry too much if you're not familiar with these notions, but it would be a failure on my part not to mention how to think about this action in a somewhat more sophisticated way.

The maps we've introduced don't just induce maps on cohomology; they induce functors between any category of sheaves satisfying Grothendieck's six-functor formalism. Most importantly on:

- constructible derived categories on $\mathrm{Gr}(k, S)$ (in any formalism you like: analytic, étale \mathbb{Q}_ℓ -sheaves, etc.) or
- derived categories of D-modules on $\mathrm{Gr}(k, S)$.

Of course, any one who knows the Riemann-Hilbert correspondence will tell you these are essentially the same, but it's nice to have both.

Upgrade to sheaves

In either case, we let

$$\mathcal{E}^k(\mathcal{M}) = (\pi_1)_* \pi_2^* \mathcal{M}[k-1] \quad \mathcal{F}^k(\mathcal{M}) = (\pi_2)_* \pi_1^* \mathcal{M}[d-k-1]$$

$$\text{Gr}(k, S) \xleftarrow{\pi_1} \text{Gr}(k \subset k+1, S) \xrightarrow{\pi_2} \text{Gr}(k+1, S)$$

Theorem (Beilinson-Lusztig-Macpherson)

For constructible sheaves or D-modules, we have

$$\mathcal{E}^{k+1} \mathcal{F}^k \cong \mathcal{F}^{k-1} \mathcal{E}^k \oplus \text{id}[d-2k] \oplus \text{id}[d-2k-2] \oplus \cdots \oplus \text{id}[2k-d] \quad (k \leq d/2)$$

$$\mathcal{E}^{k+1} \mathcal{F}^k \oplus \text{id}[2k-d] \oplus \text{id}[d-2k-2] \oplus \cdots \oplus \text{id}[d-2k] \cong \mathcal{F}^{k-1} \mathcal{E}^k \quad (k \geq d/2).$$

Obviously, this is the “same” result we got for modules over the cohomology, but if there’s anything modern mathematics has taught me, it’s that you have to be really careful about what “the same” means.

Weak \mathfrak{sl}_2 actions

Still, now that we have two examples, we can make a definition.

Definition

A weak \mathfrak{sl}_2 -action on an additive category \mathcal{C} consists of:

- A direct sum decomposition $\mathcal{C} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{C}_n$.
- Functors

$$\mathcal{E}^n : \mathcal{C}_n \rightarrow \mathcal{C}_{n+2} \quad \mathcal{F}^n : \mathcal{C}_n \rightarrow \mathcal{C}_{n-2}$$

such that satisfy the relations

$$\mathcal{E}^{n-1} \mathcal{F}^n \cong \mathcal{F}^{n+1} \mathcal{E}^n \oplus \text{id}^{\oplus n} \quad (n \geq 0)$$

$$\mathcal{E}^{n-1} \mathcal{F}^n \oplus \text{id}^{\oplus -n} \cong \mathcal{F}^{n+1} \mathcal{E}^n \quad (n \leq 0)$$

(Chuang and Rouquier would require \mathcal{E}^n and \mathcal{F}^{n+2} to be adjoint; I won't bother with that for now.)

Weak \mathfrak{sl}_2 actions

Examples:

- The first example we gave is $\mathcal{C}_{d-2k}^d \cong H_k$ -mod. There are variations of this: you might only want projective modules, or various other natural subcategories.
- Alternately, you can take $\mathcal{D}_n^d \cong \mathbb{C}$ -mod for $n \in d, d-2, \dots, -d+2, -d$ and 0 otherwise. In this case, you should take $\mathcal{E}^{-d+2k}(\mathbb{C}) \cong \mathbb{C}^{k+1}$ and $\mathcal{F}^{d-2k}(\mathbb{C}) \cong \mathbb{C}^{k+1}$.

We'll get into a lot more examples later; as you might guess, there's a notion of strong action which we'll want to discuss first.

Symmetric groups

Fix a field \mathbb{k} , and consider the group algebras of the symmetric groups S_n over \mathbb{k} . These are related by induction and restriction functors. Let $X_m = \sum_{i < m} (im)$ by the m th Jucys-Murphy element.

Proposition

The action of X_m on $\text{Res}_{S_{m-1}}^{S_m} M$ commutes with the S_{m-1} -action. That is, it is a natural transformation of the functor $\text{Res}_{S_{m-1}}^{S_m}$.

Fix $a \in \mathbb{k}$, and let

$$\mathcal{E}_a(M) = \{m \in \text{Res}_{S_{m-1}}^{S_m} M \mid (X_m - a)^N m = 0 \text{ for } N \gg 0\}.$$

$$\mathcal{F}_a(M) = \{s \otimes m \in \text{Ind}_{S_{m-1}}^{S_m} M \mid s(X_m - a)^N \otimes m = 0 \text{ for } N \gg 0\}.$$

Proposition (Lascoux-Leclerc-Thibon; Chuang-Rouquier)

For each $a \in \mathbb{k}$, the functors \mathcal{E}_a and \mathcal{F}_a define a weak \mathfrak{sl}_2 -action; under these functors, \mathbb{k} as S_0 -module “generates” all S_n -modules.

Weak \mathfrak{sl}_2 actions

In order to talk in a clear way about categorification, we need the notion of a **Grothendieck group**. There are a few variations on this notion.

Definition

Given an additive category \mathcal{C} , the **split Grothendieck group** (sGG) $K(\mathcal{C})$ of \mathcal{C} is the abelian group generated by symbols $[C]$ for C an object in \mathcal{C} , modulo the relation:

$$[C_1 \oplus C_2] = [C_1] + [C_2].$$

Let $K_{\mathbb{C}}(\mathcal{C}) \cong K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem

If \mathcal{C} is an additive category with a weak \mathfrak{sl}_2 -action, the maps

$$[E]: K_{\mathbb{C}}(\mathcal{C}) \rightarrow K_{\mathbb{C}}(\mathcal{C}) \quad [F]: K_{\mathbb{C}}(\mathcal{C}) \rightarrow K_{\mathbb{C}}(\mathcal{C})$$

generate an \mathfrak{sl}_2 -action on $K_{\mathbb{C}}(\mathcal{C})$.

Weak \mathfrak{sl}_2 actions

Theorem

Both the categories of projective H_k -modules, and the example \mathcal{D}_n^d we discussed have $K_{\mathbb{C}}$ given by the unique irreducible \mathfrak{sl}_2 representation of dimension $d + 1$.

On the other hand, if we take $d = 2$, and consider *all* H_k -modules, we'll categorify $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Philosophically, this theorem is a bit problematic. If there are a bunch of different ways to categorify the simple modules, then where are we? How will we ever get control over the general situation? For me, this is a big strike against the “weak” definition.

In fact, examples like H_k -mod show up a lot more often than ones like \mathcal{D}_n^d ; thus, we'll want a better definition that rules out the latter.

Next time...

There's an even more basic philosophical issue here: you *never* want a definition just saying things are isomorphic. You want to have control about what the isomorphisms are and what relations they satisfy.

All the hints we need about how to do this are hiding in the Grassmannians, but we're going to have to look deeper to find them.

Next time we'll get to the definition and examples of strong \mathfrak{sl}_2 -actions. This will show us how rule out \mathcal{D}^d , and get a better behaved theory of categorical actions.