

Higher representation theory in algebra and geometry: Lecture II

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References

For this lecture, useful references include:

- Chuang and Rouquier, *Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification* (introduces notion of strong \mathfrak{sl}_2 categorification and covers several examples, such as the symmetric group)
- Lauda, *A categorification of quantum $\mathfrak{sl}(2)$ and Categorized quantum $\mathfrak{sl}(2)$ and equivariant cohomology of iterated flag varieties* (discusses the relationship with the cohomology of Grassmannians and is the source of the pictures)

The slides for the talk are on my webpage at:

<http://people.virginia.edu/~btw4e/lecture-2.pdf>

Reminder about last time

So, last time we discussed the Lie algebra \mathfrak{sl}_2 , its quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ and how we might categorify them.

The underlying idea was to look at these objects in a way that makes the categorification of them very natural, in connection with Grassmannians.

In particular, the generators E and F were connected to the diagram

$$\text{Gr}(k, S) \xleftarrow{\pi_1} \text{Gr}(k \subset k+1, S) \xrightarrow{\pi_2} \text{Gr}(k+1, S)$$

We first interpreted this in terms of counting points, but then decided we could “upgrade” from this to considering cohomology or push and pull for sheaves on this diagram.

Cohomology and bimodules

$$H_k := H^*(\mathrm{Gr}(k, S))$$

$${}_{k-1}H_k := H^*(\mathrm{Gr}(k-1 \subset k, S))(k-1) \quad {}_{k+1}H_k := H^*(\mathrm{Gr}(k+1 \supset k, S))(d-k-1)$$

$$\mathcal{E}^k(M) := {}_{k-1}H_k \otimes_{H_k} M \quad \mathcal{F}^k(M) := {}_{k+1}H_k \otimes_{H_k} M.$$

Theorem

$$\mathcal{E}^{k+1} \mathcal{F}^k \cong \mathcal{F}^{k-1} \mathcal{E}^k \oplus \mathrm{id}^{\oplus [d-2k]_q} \quad (k \leq d/2)$$

$$\mathcal{E}^{k+1} \mathcal{F}^k \oplus \mathrm{id}^{\oplus [2k-d]_q} \cong \mathcal{F}^{k-1} \mathcal{E}^k \quad (k \geq d/2)$$

So, this is the relations of (quantum) \mathfrak{sl}_2 in a categorified form. But as a definition, we decided this was unsatisfying; we wouldn't know if we found other functors satisfying the same isomorphisms if we were really seeing the same thing.

Natural transformations

So, the thing that's missing is some understanding of what these isomorphisms are, but more generally of maps between functors.

There's an obvious set of questions we haven't looked at at all: If I have two monomials in the functors \mathcal{F} and \mathcal{E} , what are the natural transformations between them?

If you don't like natural transformations, you can just think of these as maps between the tensor products of bimodules.

Cohomology of Grassmannians and partial flag varieties

If we want to get our hands dirty, we'll have to actually think a bit about the cohomology of Grassmannians.

Of course, the Grassmannian carries a tautological vector bundle $T \subset \mathbf{Gr}(k, S) \times S$ whose fiber over a point is the vector space itself; there's also the quotient S/T of the trivial bundle with fiber S by T .

This gives us a bunch of cohomology classes, by taking the Chern classes of these bundles

$$x_i = c_i(T) \quad w_i = c_i(S/T).$$

By the Whitney sum formula, these satisfy the relation

$$(1 + c_1(T)t + c_2(T)t^2 + \cdots)(1 + c_1(S/T)t + c_2(S/T)t^2 + \cdots) = 1 + c_1(S)t + \cdots$$

Cohomology of Grassmannians and partial flag varieties

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$$x_i = c_i(T) \quad w_i = c_i(S/T).$$

By the Whitney sum formula, these satisfy the relation

$$(1 + x_1t + x_2t^2 + \cdots)(1 + w_1t + w_2t^2 + \cdots) = 1.$$

Cohomology of Grassmannians and partial flag varieties

So, for example:

$$1 = 1$$

You can solve for x_i 's in terms of w_i 's or vice versa.

Cohomology of Grassmannians and partial flag varieties

So, for example:

$$x_1 + w_1 = 0$$

You can solve for x_i 's in terms of w_i 's or vice versa.

Cohomology of Grassmannians and partial flag varieties

So, for example:

$$x_2 + x_1 w_1 + w_2 = 0$$

You can solve for x_i 's in terms of w_i 's or vice versa.

Cohomology of Grassmannians and partial flag varieties

So, for example:

$$x_3 + x_2w_1 + x_1w_2 + w_3 = 0$$

You can solve for x_i 's in terms of w_i 's or vice versa.

Cohomology of Grassmannians and partial flag varieties

So, for example:

$$x_3 + x_2w_1 + x_1w_2 + w_3 = 0$$

You can solve for x_i 's in terms of w_i 's or vice versa.

Theorem

The cohomology $H^(\mathrm{Gr}(k, S))$ of the Grassmannian is the quotient of the polynomial ring $\mathbb{C}[x_1, \dots, x_k, w_1, \dots, w_{d-k}]$ by these Whitney sum relations.*

Alternatively, you can consider an infinite polynomial ring $\mathbb{C}[x_1, \dots, w_1, \dots]$ modulo the Whitney sum relation, and write all the cohomologies of all the Grassmannians as quotients of this by setting

$$x_i = 0 \quad (i > k) \qquad w_j = 0 \quad (j > d - k).$$

Natural transformations

This also handled the question of one set of natural transformations:

Proposition

The natural transformations of the identity functor id^k are the ring $H^(\text{Gr}(k, S))$ itself.*

OK, that was easy. What about \mathcal{E}^k or \mathcal{F}^k ?

The ring $H^*(\text{Gr}(k \subset k+1, S))$ can be found in a very similar way. But now, we have k and $k+1$ dimensional tautological bundles T^k, T^{k+1} , and a line bundle T^{k+1}/T^k .

The notation can get confusing, since I have a class I was calling x_i on $H^*(\text{Gr}(k, S))$ and on $H^*(\text{Gr}(k+1, S))$, and these don't pullback to the same place: one goes to $c_i(T^k)$ and the other to $c_i(T^{k+1})$.

Cohomology and symmetric polynomials

In the interest of not going crazy, let's look at this in a different way: let z_1, \dots, z_k be the Chern roots of T^k , and z_{k+1}, \dots, z_d be the Chern roots of S/T^k . For us these are just formal symbols, which satisfy

$$e_i(1, k) = e_i(z_1, \dots, z_k) = c_i(T^k) \quad e_i(k+1, d) = e_i(z_{k+1}, \dots, z_d) = c_i(S/T^k)$$

In the formulation with Chern roots, the Whitney sum relations simply say that the Chern roots of S are the union of those for T^k and S/T^k . The rest follows from the fact that:

$$e_i(1, d) = e_i(1, k) + e_{i-1}(1, k)e_1(k+1, d) + \dots + e_i(k+1, d).$$

The Whitney sum relations for the Grassmannian simply say that $e_i(1, d) = 0$ for all $i > 0$.

Cohomology and symmetric polynomials

Definition

The **coinvariant algebra** C_d is the quotient of $\mathbb{C}[z_1, \dots, z_d]$ by the relations $e_i(1, d) = 0$. This inherits an S_d action permuting the variables.

Let $S_{m_1, \dots, m_p} := S_{m_1} \times \dots \times S_{m_p} \subset S_d$, and $C_d^{k_1, \dots, k_p} = C_d^{S_{k_1, k_2 - k_1, k_3 - k_2, \dots, d - k_p}}$.

Proposition

The algebra $H^*(\mathrm{Gr}(k, S))$ is isomorphic to C_d^k which is generated by the elementary symmetric functions $e_i(1, k)$ and $e_i(k + 1, d)$.

This can be generalized to the partial flag variety $\mathrm{Gr}(k_1 \subset k_2 \subset \dots \subset k_p)$.

Proposition

The cohomology $\mathrm{Gr}(k_1 \subset k_2 \subset \dots \subset k_p)$ is isomorphic to $C_d^{k_1, k_2, \dots, k_p}$, generated by $e_i(1, k_1), e_i(k_1 + 1, k_2), \dots, e_i(k_p + 1, d)$ in C_d .

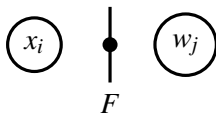
Cohomology and symmetric polynomials

Thus, the bimodule ${}_{k+1}H_k$ can be thought of as $C_d^{k,k+1}$, with the left action by $H_{k+1} = C_d^{k+1}$ and the right action by $H_k = C_d^k$.

Proposition

Since H_{k+1} and H_k generate $C_d^{k,k+1}$, the bimodule maps from ${}_{k+1}H_k$ to itself are given by multiplication by elements of $C_d^{k,k+1}$.

Visually, we can represent these elements using diagrams:



The left/right sides correspond to the left/right action, the bottom to the source bimodule and the top to the target. I represent the action of z_{k+1} by a dot on the line.

Diagrams

These diagrams have the huge advantage that I don't confuse the left and right actions. I can write a relation like $c_i(T^{k+1}) = c_i(T^k) + c_1(T^{k+1}/T^k)c_i(T^k)$:

$$\begin{array}{c} \textcircled{x_i} \\ | \\ F \end{array} = \begin{array}{c} | \\ | \\ F \end{array} \textcircled{x_i} + \begin{array}{c} | \\ \bullet \\ | \\ F \end{array} \textcircled{x_{i-1}}$$

without making a mess separating the left and right actions.

In general, I can describe any element of a tensor product of ${}_{k\pm 1}H_k$'s using a diagram like this, with a line for each term in the tensor product.

$$\begin{array}{c} \textcircled{x_i} \\ | \\ \bullet \\ | \\ F \end{array} \textcircled{w_j} \begin{array}{c} | \\ | \\ E \end{array} \begin{array}{c} | \\ | \\ \bullet \\ | \\ F \end{array}$$

More natural transformations

This describes all elements, but not all bimodule maps.

For example, if $d = 2$, then ${}_2H_1 \otimes {}_1H_0 \cong \mathbb{C}[t]/(t^2) \otimes_{\mathbb{C}[t]/(t^2)} \mathbb{C}[t]/(t^2) \cong \mathbb{C}^2$, so we must get the matrices acting on here somehow.

In general, ${}_{k+2}H_{k+1} \otimes {}_{k+1}H_k$ is always two copies of the same module.

$$\begin{array}{ccc} & \text{Gr}(k \subset k+1 \subset k+2, S) & \\ & \downarrow \pi_{1,3} & \\ \text{Gr}(k, S) & \xleftarrow{\pi_1} \text{Gr}(k \subset k+2, S) \xrightarrow{\pi_2} & \text{Gr}(k+2, S) \end{array}$$

The map $\pi_{1,3}$ is a fiber bundle with fiber \mathbb{P}^1 . Thus as a bimodule:

$${}_{k+2}H_{k+1} \otimes {}_{k+1}H_k \cong H^*(\text{Gr}(k \subset k+1 \subset k+2, S)) \cong H^*(\text{Gr}(k \subset k+2, S))^{\oplus 2}$$

Demazure operators

From the perspective of polynomials, this is also easy to see: if I consider the invariants of $C_d^{k,k+1,k+2}$, this has a natural action of $(k+1, k+2)$.

Proposition

The invariants and anti-invariants of $(k+1, k+2)$ are both free modules of rank 1 over $C_d^{k,k+2} \cong H^(\text{Gr}(k \subset k+2, S))$.*

Thus, my bimodule maps as a ring are a 2×2 matrix algebra over $C_d^{k,k+2}$.

Definition

Let the **Demazure operator** $D_i: C_d \rightarrow C_d$ be the map

$$D_i(f) = \frac{f(z_1, \dots, z_{i+1}, z_i, \dots, z_d) - f(z_1, \dots, z_d)}{z_{i+1} - z_i}$$

This kills polynomials invariant under $(i, i+1)$, and thus is well-defined.

Demazure operators

Geometrically, I can think of the Demazure operator as integration along the projection map $\mathrm{Gr}(k \subset k+1 \subset k+2, S) \rightarrow \mathrm{Gr}(k \subset k+2, S)$.

Important facts:

- $D_i(f) = 0$ if $f = f^{(i,i+1)}$
- $D_i(fg) = D_i(f)g + f^{(i,i+1)}D_i(g)$. Thus, D_i is a module map for $C_d^{i-1,i+1}$.
- $D_i(fg) = fD_i(g)$ if $f = f^{(i,i+1)}$
- $D_i(x_i f) - x_{i+1}D_i(f) = x_i D(f) - D_i(x_{i+1}f) = f$, since $D_i(x_i) = 1, D_i(x_{i+1}) = -1$.
- D_i is the unique map satisfying these properties.

Demazure operators

Theorem

Multiplication by $C_d^{k,k+1,k+2}$ and the Demazure operator D_{k+1} generate bimodule endomorphisms of ${}_{k+2}H_{k+1} \otimes {}_{k+1}H_k \cong C_d^{k,k+1,k+2}$.

In terms of pictures, I'll draw a Demazure operator as a crossing:



My relations then become:

Theorem

These relations give a presentation of $\text{End}({}_{k+2}H_{k+1} \otimes {}_{k+1}H_k)$.

Demazure operators

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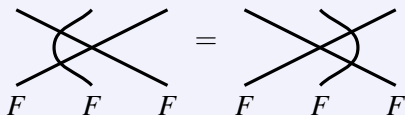
These relations give a presentation of $\text{End}({}_{k+2}H_{k+1} \otimes {}_{k+1}H_k)$.

Demazure operators

OK, let's keep going; what about the triple tensor product? This is the endomorphisms of the module $C_d^{k,k+1,k+2,k+3}$.

Here, the Demazure operators D_{k+1} and D_{k+2} both commute with $C_d^{k,k+3}$, and satisfy the relation:

$$D_{k+1}D_{k+2}D_{k+1}(f) = D_{k+2}D_{k+1}D_{k+2}(f)$$



Theorem

The endomorphisms of the functor $\mathcal{F}^{k+\ell} \dots \mathcal{F}^k$ are generated by the Demazure operators and multiplication by elements of $C_d^{k,k+1,\dots,k+\ell}$; the relations these satisfy are the ones we know.

The nilHecke algebra

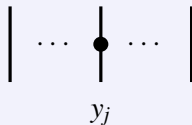
Thus, on $\mathcal{F}^{k+\ell} \dots \mathcal{F}^k$, there's a natural algebra that acts:

Definition

The **nilHecke algebra** N_ℓ is the algebra generated by symbols $\psi_1, \dots, \psi_{\ell-1}, y_1, \dots, y_\ell$ subject to the relations:

$$\psi_i y_i - y_{i+1} \psi_i = y_i \psi_i - \psi_i y_{i+1} = 1 \quad \psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1}$$

$$\psi_i \psi_j = \psi_j \psi_i \quad (|i - j| \neq 1) \quad y_i y_j = y_j y_i$$



The nilHecke algebra

The nilHecke algebra has a very natural geometric realization: let $\mathcal{F}l(\ell)$ be the ℓ -step complete flag variety.

We have 3 projections $\pi_{1,2}, \pi_{1,3}, \pi_{2,3}: \mathcal{F}l(\ell)^3 \rightarrow \mathcal{F}l(\ell)^2$.

We can endow $H_*^{GL_\ell}(\mathcal{F}l(\ell) \times \mathcal{F}l(\ell))$ with a convolution multiplication by

$$A \star B = (\pi_{1,3})_*(\pi_{1,2}^*A \cap \pi_{2,3}^*B).$$

Proposition

We have a canonical isomorphism between the nilHecke algebra and $H_^{GL_\ell}(\mathcal{F}l(\ell) \times \mathcal{F}l(\ell))$ endowed with the convolution multiplication.*

The polynomial generators y_i come from the equivariant homology of $\mathcal{F}l(\ell)_\Delta$; the classes ψ_i correspond to the fiber product

$$\mathcal{F}l(\ell) \times_{\text{Gr}(1 \subset \dots \subset i-1 \subset i+1 \subset \dots \subset \ell)} \mathcal{F}l(\ell) \subset \mathcal{F}l(\ell) \times \mathcal{F}l(\ell).$$

The nilHecke algebra

The other way to think about the nilHecke algebra N_ℓ is its natural representation on $\mathbb{k}[z_1, \dots, z_\ell]$ via multiplication and Demazure operators

$$y_i \cdot f = z_i f \quad \psi_i \cdot f = D_i(f).$$

Theorem

This action induces an isomorphism

$$N_\ell \cong \text{End}_{\mathbb{k}[z_1, \dots, z_\ell]^{S_\ell}}(\mathbb{k}[z_1, \dots, z_\ell]) \cong M_{\ell!}(\mathbb{k}[z_1, \dots, z_\ell]^{S_\ell}).$$

In particular, N_ℓ and $\mathbb{k}[z_1, \dots, z_\ell]^{S_\ell}$ are Morita equivalent, and N_ℓ has a unique simple module C_ℓ which is $\ell!$ -dimensional.

It's useful for us to note that

$$e_\ell = \psi_1 \psi_2 \psi_1 \psi_3 \psi_2 \psi_1 \cdots \psi_\ell \psi_{\ell-1} \cdots \psi_1 y_1^{\ell-1} y_2^{\ell-2} \cdots y_{\ell-1}$$

is a primitive idempotent. Thus, for any N_ℓ module, $M \cong (e_\ell M)^{\oplus \ell!}$.

Symmetric groups

Fix a field \mathbb{k} , and consider the group algebras of the symmetric groups S_n over \mathbb{k} . These are related by induction and restriction functors. Let $X_m = \sum_{i < m} (im)$ be the m th Jucys-Murphy element.

Proposition

The action of X_m on $\text{Res}_{S_{m-1}}^{S_m} M$ commutes with the S_{m-1} -action. That is, it is a natural transformation of the functor $\text{Res}_{S_{m-1}}^{S_m}$.

Fix $a \in \mathbb{k}$, and let

$$\mathcal{E}_a(M) = \{h \in \text{Res}_{S_{m-1}}^{S_m} M \mid (X_m - a)^N h = 0 \text{ for } N \gg 0\}.$$

$$\mathcal{F}_a(M) = \{s \otimes h \in \text{Ind}_{S_{m-1}}^{S_m} M \mid s(X_m - a)^N \otimes h = 0 \text{ for } N \gg 0\}.$$

Symmetric groups

In this case, \mathcal{F}_a^2 carries an action of N_2 : there are natural transformations given by s_{n+1}, X_{n+1} and X_{n+2} .

We can use these to define the operators ψ_1, y_1 and y_2 by the formulae:

$$y_1 = X_{n+1} - a \quad y_2 = X_{n+2} - a$$

$$\psi_1 = (s_{n+1} + 1)(1 + (X_{i+1} - X_i) + (X_{i+1} - X_i)^2 + \dots)$$

You can have some fun checking the relations.

This extends to an action of N_ℓ on \mathcal{F}_a^ℓ .

OK, so that's at least two cases where a nilHecke action comes in. So maybe this is a universal feature of good \mathfrak{sl}_2 -actions?

Strong \mathfrak{sl}_2 -actions

Definition

A **strong \mathfrak{sl}_2 -action** on an additive category \mathcal{C} consists of:

- A direct sum decomposition $\mathcal{C} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{C}_n$.
- Adjoint functors

$$\mathcal{F}: \mathcal{C}_{n+2} \rightarrow \mathcal{C}_n \quad \mathcal{E}: \mathcal{C}_n \rightarrow \mathcal{C}_{n+2}$$

such that satisfy the relations on \mathcal{C}_n

$$\mathcal{E}\mathcal{F} \cong \mathcal{F}\mathcal{E} \oplus \text{id}^{\oplus n} \quad (n \geq 0)$$

$$\mathcal{E}\mathcal{F} \oplus \text{id}^{\oplus -n} \cong \mathcal{F}\mathcal{E} \quad (n \leq 0)$$

- Compatible actions of the nilHecke algebra N_ℓ on \mathcal{F}^ℓ for every n, ℓ , induced by nilpotent operators $y: \mathcal{F} \rightarrow \mathcal{F}$ and $\psi: \mathcal{F}^2 \rightarrow \mathcal{F}^2$.

I'm also going to assume **integrability**:

- For every object M , we have $\mathcal{F}^\ell M = \mathcal{E}^\ell M = 0$ for $\ell \gg 0$.

Strong \mathfrak{sl}_2 -actions

Examples:

- $\mathcal{C}_{d-2k} = H_k$ -mod: we've shown all the properties except the adjunction of \mathcal{E} and \mathcal{F} ; this is the statement that

$$\mathrm{Hom}_{H_k}({}_{k\pm 1}H_k, H_k) \cong {}_kH_{k\pm 1}.$$

This map comes from relative Poincaré duality for the map $\mathrm{Gr}(k \subset k+1, S) \rightarrow \mathrm{Gr}(k, S)$.

- $\bigoplus_n \mathcal{C}_n = \bigoplus_m \mathbb{k}S_m$ -mod with functors $(\mathcal{E}_a, \mathcal{F}_a)$: We've checked all the conditions here, except $[E, F]$ equation (and I never told you how to weight things)....

A mixed blessing of this area is that the definitions are very flexible. There are lots of different ways of saying them, so you can weaken them when trying to do something that has a strong action, or strengthen them when trying to get consequences.

Strong \mathfrak{sl}_2 -actions

Theorem (Chuang-Rouquier)

Assume \mathcal{C} is abelian and \mathcal{E} and \mathcal{F} exact. Then the $[E, F]$ relations hold if for each simple L ,

$$[L : \mathcal{E}\mathcal{F}M] = [L : \mathcal{F}\mathcal{E}M] + n[L : M].$$

for some set of objects $M \in \mathcal{C}_n$ which you can filter any projective with.

Thus, one can check this equation for M simple, but there are other interesting possibilities, like Specht modules for symmetric groups.

The Specht modules S_λ are a collection of representations of $\mathbb{Z}S_n$ indexed by partitions of m . If $\text{char}(\mathbb{k}) = 0$, then $\mathbb{k} \otimes_{\mathbb{Z}} S_\lambda$ are a complete set of irreducible modules; in general, every projective $\mathbb{k}S_m$ module has a Specht filtration.

Symmetric groups

Given a partition λ , we fill its box at (i, j) with its content $j - i \in \mathbb{k}$.

The weight of a Specht module for $(\mathcal{E}_a, \mathcal{F}_a)$ is

$$\#\{\text{boxes of content } a \pm 1\} - 2\#\{\text{boxes of content } a\} + \delta_{a,0}$$

So if $\mathbb{k} = \mathbb{F}_3$ and $a = 0$, then the partition $(3, 1)$ has weight 2:

2		
0	1	2

The module $\mathcal{E}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$ (resp. $\mathcal{F}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$) has a filtration whose subquotients are $\mathbb{k} \otimes_{\mathbb{Z}} S_{\mu}$ where μ is λ minus (resp. plus) a box of content a .

The $[E, F]$ relations follow from counting Specht multiplicities on both sides of the equation.

Symmetric groups

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The weight of a Specht module for $(\mathcal{E}_a, \mathcal{F}_a)$ is

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So if $\mathbb{k} = \mathbb{F}_3$ and $a = 1$, then the partition $(3, 1)$ has weight 1:

2		
0	1	2

The module $\mathcal{E}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$ (resp. $\mathcal{F}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$) has a filtration whose subquotients are $\mathbb{k} \otimes_{\mathbb{Z}} S_{\mu}$ where μ is λ minus (resp. plus) a box of content a .

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So if $\mathbb{k} = \mathbb{F}_3$ and $a = 2$, then the partition $(3, 1)$ has weight -2 :

2		
0	1	2

The module $\mathcal{E}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$ (resp. $\mathcal{F}_a(\mathbb{k} \otimes_{\mathbb{Z}} S_{\lambda})$) has a filtration whose subquotients are $\mathbb{k} \otimes_{\mathbb{Z}} S_{\mu}$ where μ is λ minus (resp. plus) a box of content a .

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Representations of \mathfrak{gl}_n

Let \mathcal{B} be an Artinian category of representations of \mathfrak{gl}_m (or even the superalgebra $\mathfrak{gl}_{m|q}$) which is closed under tensor product with finite dimensional modules (finite dimensionals, parabolic category \mathcal{O} , etc.).

Definition

Let $\Omega = \sum e_{i,j} \otimes e_{j,i} \in \mathfrak{gl}_n \otimes \mathfrak{gl}_n$.

$$\mathcal{E}_a(M) := \{g \in M \otimes \mathbb{C}^m \mid (\Omega - a)^p g = 0 \text{ for } p \gg 0\}$$

$$\mathcal{F}_a(M) := \{g \in M \otimes (\mathbb{C}^m)^* \mid (\Omega - a)^p g = 0 \text{ for } p \gg 0\}$$

This has a N_ℓ -action on \mathcal{F}_a^ℓ much like that for symmetric groups:

$$y = \Omega - a \quad \psi = (1 \otimes s)(1 - \Omega_{13} + \Omega_{13}^2 - \dots)$$

Theorem

The category \mathcal{B} has a strong categorical \mathfrak{sl}_2 action by $\mathcal{E}_a, \mathcal{F}_a$; the weight is a function of how higher Casimirs act on a module.

Divided powers

I hope you're now somewhat convinced that this isn't a totally vacuous notion. But it's not a useful one if it doesn't have consequences.

Assume from now on that $\mathcal{C} = \bigoplus_n \mathcal{C}_n$ is a category with strong \mathfrak{sl}_2 action by functors $\mathcal{E}, \mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}$.

The first consequence comes from the fact that N_ℓ is a matrix algebra.

Definition

Let $\mathcal{E}^{(\ell)} = e_\ell \mathcal{E}^\ell$ and $\mathcal{F}^{(\ell)} = e_\ell \mathcal{F}^\ell$. We have that

$$\mathcal{E}^\ell \cong (\mathcal{E}^{(\ell)})^{\oplus \ell!} \quad \mathcal{F}^\ell \cong (\mathcal{F}^{(\ell)})^{\oplus \ell!}$$

Thus, these functors categorify the divided powers $E^{(\ell)} := E^\ell / (\ell!)$ and $F^{(\ell)} := F^\ell / (\ell!)$.

Uniqueness

We have a commutation formula

$$E^{(\ell)} F^{(\ell)} = \sum_{j=0}^{\ell} F^{(j)} \frac{((H - 2j)(H - 2j - 1) \cdots (H - \ell - j + 1))}{(\ell - j)!} E^{(j)}.$$

In particular, if v is a highest weight vector of weight d , then

$$E^{(d)} F^{(d)} v = \frac{H(H - 1) \cdots (H - d + 1)}{d!} v = v.$$

Thus, if M is an object in \mathcal{C}_d such that $\mathcal{E}M = 0$, then $\mathcal{E}^{(d)} \mathcal{F}^{(d)} M \cong M$.

Proposition

In this case, we have a natural isomorphism $\text{End}(M) \cong \text{End}(\mathcal{F}^{(d)} M)$

Uniqueness

These divided powers help us to work out the structure of a categorification.

Theorem

Assume that M is an object in \mathcal{C}_d such that $\text{End}(M) \cong \mathbb{k}$ and $\mathcal{E}M = 0$. Then $\text{End}(\mathcal{F}^{(k)}M) \cong H_k \cong H^(\text{Gr}(k, \mathbb{C}^d); \mathbb{k})$; in particular, $\mathcal{F}^{(k)}M$ is indecomposable.*

Steps in the proof:

- Note that $\text{End}(\mathcal{F}^k M) \rightarrow \text{End}(\mathcal{F}^d M) \cong \text{End}_{\mathbb{k}}(\mathcal{C}_d)$ must be injective. This shows that no class in $\mathbb{k}[y_1, \dots, y_k]^{S_k}$ that goes to 0 in $\text{End}(\mathcal{F}^k M)$ can go to 0 in $H^*(\text{Gr}(k, \mathbb{C}^d); \mathbb{k})$.
- On the other hand,

$$\dim \text{End}(\mathcal{F}^{(k)}M) = \dim \text{Hom}(M, \mathcal{E}^{(k)}\mathcal{F}^{(k)}M) = \binom{d}{k}.$$

This is only possible if the map $\mathbb{k}[y_1, \dots, y_k]^{S_k} \rightarrow \text{End}(\mathcal{F}^{(k)}M)$ induces the desired isomorphism.

Uniqueness

This is one version of a uniqueness theorem:

Theorem (Chuang-Rouquier)

If \mathcal{C} has Grothendieck group isomorphic to $V_d \cong \text{Sym}^{d-1}(\mathbb{C}^2)$, and $\mathcal{C}_d \cong \mathbb{k}\text{-mod}$, then \mathcal{C}_{d-2k} is equivariantly equivalent to H_k -proj.

You might rightly protest that you might have $\mathcal{C}_d \not\cong \mathbb{k}\text{-mod}$; this is an issue that's easily fixed, though:

We have a natural map from $\mathbb{k}[y_1, \dots, y_d]^{S_d} \rightarrow \text{End}(\mathcal{F}^{(d)}M) \cong \text{End}(M)$; the more general version of this theorem is that

$$\text{End}(\mathcal{F}^{(k)}M) \cong \mathbb{k}[y_1, \dots, y_d]^{S_{k,d-k}} \otimes_{\mathbb{k}[y_1, \dots, y_d]^{S_d}} \text{End}(M).$$

Next time...

So, next time, we'll talk about 2-categories, how to fix the $[E, F]$ relation, and other issues of tightening the definition, and discuss the marquee application of this theory: the proof of the Broué conjecture for symmetric groups.

Like last time, we've get up the pieces set up on the board, but we have to wait until next time to take the king.