

Higher representation theory in algebra and geometry: Lecture III

Ben Webster

UVA

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References

For this lecture, useful references include:

- Chuang and Rouquier, *Derived equivalences for symmetric groups and \mathfrak{sl}_2 -categorification* (first applies principle of reducing to minimal categorifications, and constructs derived equivalences lifting Weyl group elements)
- Rouquier, *2-Kac-Moody algebras* (really runs with this principle, and applies it in some new ways)
- Lauda, *A categorification of quantum $\mathfrak{sl}(2)$ and Categorified quantum $\mathfrak{sl}(2)$ and equivariant cohomology of iterated flag varieties* (goes much more heavily to into 2-category philosophy and diagrammatics)

The slides for the talk are on my webpage at:

<http://people.virginia.edu/~btw4e/lecture-3.pdf>

You can also find some proofs that I didn't feel like going through in class at:

https://pages.shanti.virginia.edu/Higher_Rep_Theory/

Reminder about last time

Last time, we talked about the definition of a strong categorical \mathfrak{sl}_2 action (from now on, I won't bother saying “strong”).

This combined the “obvious” conditions for a weak categorical \mathfrak{sl}_2 -action with an adjunction, and an action of the nilHecke algebra.

We saw that once we imposed these conditions, there was only one way of categorifying a given simple module,

$$\mathcal{A}_{d-2k}^d = H^*(\mathrm{Gr}(k, \mathbb{C}^d); \mathbb{k}).$$

So, seems like a promising approach. However, to really understand this picture, we'll want to think about it a bit differently.

A 2-category

We can think of a \mathfrak{sl}_2 -action as a representation of a 2-category.

Definition

A **2-category** is a category whose morphism spaces are themselves categories. That is, we have objects, morphisms and morphisms between morphisms (usually called 2-morphisms).

The most important example of a 2-category is the set \mathbf{Cat} of categories:

- the objects are categories,
- the morphisms are functors,
- the 2-morphisms are natural transformations.

Of course, one can impose extra structure (\mathbb{k} -algebras instead of rings, etc.).

There are two ways of composing 2-morphisms: **vertical** (composing natural transformations, composing bimodule maps) and **horizontal** (composing functors, tensoring bimodules). I'll use \otimes for horizontal.

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The other key example \mathbf{Bim} is rings and bimodules:

- the objects are rings,
- the morphisms are bimodules,
- the 2-morphisms are bimodule maps.

Of course, one can impose extra structure (\mathbb{k} -algebras instead of rings, etc.).

There are two ways of composing 2-morphisms: **vertical** (composing natural transformations, composing bimodule maps) and **horizontal** (composing functors, tensoring bimodules). I'll use \otimes for horizontal.

Pictorial calculus for 2-categories

There's a close connection between 2-categories and 2-dimensional topology. One reflection of this is that it's natural to use planar diagrams to describe morphisms in 2-categories.

We can draw planar diagrams where:

- regions labeled with an object;
- curves where regions meet (assumed to have no local maxima or minima) labeled with a 1-morphism from the region on the left to that on the right;
- points where curves meet labeled with a 2-morphism from the composition of the lower strands to the upper strands.

A horizontal slice in this picture corresponds to a 1-morphism from the left-most region to the right-most composing all the 1-morphisms you pass. A horizontal strip corresponds to a 2-morphism from the bottom to the top, composing the 2-morphisms you pass. This matches vertical and horizontal composition with stacking in these directions.

Adjoints

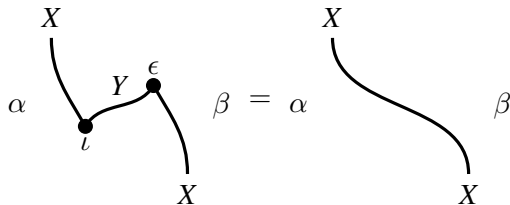
Definition

We say that 1-morphisms $X: \alpha \rightarrow \beta$ and $Y: \beta \rightarrow \alpha$ in a 2-category are adjoint if there are 2-morphisms

$$\iota: \text{id}_\alpha \rightarrow YX \quad \epsilon: XY \rightarrow \text{id}_\beta$$

such that $(\epsilon \otimes 1_X)(1_X \otimes \iota) = 1_X$ and $(1_Y \otimes \epsilon)(\iota \otimes 1_Y) = 1_Y$

This looks horrible, but if you draw it in pictures, it's fine:



Adjoints

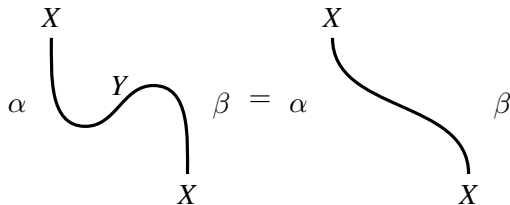
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People often smooth these particular 2-morphisms to cups and caps.

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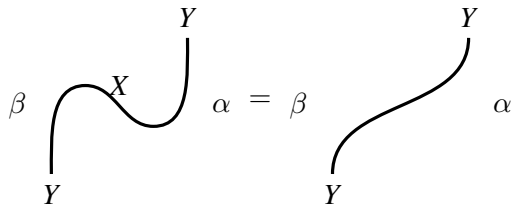
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A 2-category

Now, if I told you that an **X structure** on a ring R was a set of elements $\{a_i\}$ that satisfies relations $\{r_j(\mathbf{a})\}$, you'd say, "that's just a ring homomorphism."

Similarly, if I tell you I have a bunch of functors and natural transformations that satisfy relations, you should say "that's just a 2-functor."

Definition

Let \mathfrak{A} be the 2-category whose objects are the integers \mathbb{Z} equipped with 1-morphisms formal sums of compositions of $\mathcal{E}(n): n \rightarrow n + 2$ and $\mathcal{F}(n): n \rightarrow n - 2$ such that:

- there are 2-morphisms $y: \mathcal{F}(n) \rightarrow \mathcal{F}(n)$ and $\psi: \mathcal{F}(n-2)\mathcal{F}(n) \rightarrow \mathcal{F}(n-2)\mathcal{F}(n)$, which induce an action of N_ℓ on $\mathcal{F}(n-2\ell+2) \cdots \mathcal{F}(n)$.
- \mathcal{F} is left adjoint to \mathcal{E} (that is, there are morphisms

$$\iota: \text{id}_n \rightarrow \mathcal{E}(n-2)\mathcal{F}(n) \quad \epsilon: \mathcal{F}(n+2)\mathcal{E}(n) \rightarrow \text{id}_n$$

that satisfy adjunction relations).

A 2-category

Pictorially, we denote \mathcal{E} and \mathcal{F} with upward and downward pointing strands.



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The relations are the ones we've already seen.

$$n \quad \begin{array}{c} \downarrow \\ \curvearrowright \end{array} \quad n-2 = n \quad \begin{array}{c} \downarrow \\ \curvearrowleft \end{array} \quad n-2$$

A 2-category

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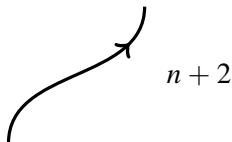
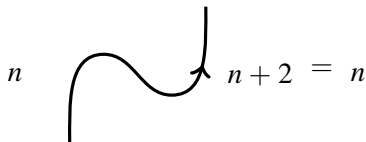


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$$\begin{array}{c} n \\ \text{diagram of } \psi \text{ with a dot on the bottom-left strand} \end{array}
 \quad
 n - 4 = n
 \quad
 \begin{array}{c} \text{diagram of } \psi \text{ with a dot on the top-right strand} \\ n - 4 + n \end{array}
 \quad
 \begin{array}{c} \downarrow \\ \text{diagram of } y \end{array}
 \quad
 \begin{array}{c} \downarrow \\ \text{diagram of } y \end{array}
 \quad
 n - 4$$

A 2-category

Pictorially, we denote \mathcal{E} and \mathcal{F} with upward and downward pointing strands.



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The relations are the ones we've already seen.

$$n \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad n - 4 = n \quad \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bullet \end{array} \quad n - 4 + n \quad \begin{array}{c} \downarrow \\ | \\ \downarrow \end{array} \quad \begin{array}{c} \downarrow \\ | \\ \downarrow \end{array} \quad n - 4$$

A 2-category

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The relations are the ones we've already seen.

$$n \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad n - 6 = n \quad \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad n - 6$$

A 2-category

Definition

A **representation** of a 2-category \mathfrak{X} is a 2-functor $\mathfrak{X} \rightarrow \mathbf{Cat}$. That is, an assignment of a category to each object, functor to each 1-morphism and natural transformation to each 2-morphism, in a way compatible with composition.

Obviously, any \mathfrak{sl}_2 action defines a representation of \mathfrak{U} , sending

$$n \mapsto \mathcal{C}_n \quad \mathcal{E} \mapsto \mathcal{E} \quad \mathcal{F} \mapsto \mathcal{F}$$

Unfortunately, we don't know how to phrase the $[E, F]$ relation in this language. It's still not categorical enough!

Checking isomorphisms

Chuang and Rouquier taught us a very valuable principle for thinking about this:

Theorem (Chuang-Rouquier)

If a 2-morphism $p: F \rightarrow G$ in \mathfrak{A} induces an isomorphism between functors in \mathcal{A}^d for every d , then it induces an isomorphism for every integrable categorical \mathfrak{sl}_2 action on a \mathbb{k} -linear category.

Just to make my life easier, I'm going to assume that my category is abelian and artinian and \mathcal{E} and \mathcal{F} are exact (in particular, that any object has a simple subobject); for most applications, this suffices.

Also, I should note now that my proof is a loose paraphrase of Chuang and Rouquier's; it will get tiresome if I ascribe every single lemma to them.

Simple modules

In order to prove this, we want to understand the structure of an arbitrary categorification, and break it into “pieces” that correspond to \mathcal{A}^d .

Lemma

Every simple S is either highest (resp. lowest) weight, or the quotient of $\mathcal{F}S'$ (resp. a submodule of $\mathcal{E}S'$) for a unique simple S' . We denote this simple $S' = fS$ (resp. $S' = eS$).

Proof.

Assume $\mathcal{E}S \neq 0$; then there must be some simple submodule $S' \subset \mathcal{E}S$. Adjunction then induces a non-zero (and thus surjective) map $\mathcal{F}S' \rightarrow S$.

Proof of uniqueness is trickier; see webpage. Most important part is that $\text{End}(\mathcal{E}^{(i)}S)$ is the cohomology of a Grassmannian. □

Kashiwara operators

Thus, the set of simple modules for any abelian categorification breaks up as a disjoint union of strings where e and f move up and down along the strings.

Lemma

Every string contains exactly one module of each weight of the form $d, d - 2, d - 4, \dots, -d + 2, -d$ for some integer d .

This follows immediately from the fact that if S is a highest weight of weight d , then $\mathcal{F}^{(d)}S \cong f^d S$ is a lowest weight simple of weight $-d$.

That is, the set of simples break into a union of weight strings that look like simple modules of \mathfrak{sl}_2 .

In more fancy terminology, this is the crystal of an integrable \mathfrak{sl}_2 module.

Kashiwara operators

For an abelian categorification \mathcal{C} , let $\mathcal{C}(< d)$ be the modules whose composition factors live in weight strings that contain no simples of weight $\geq d$.

Lemma

If L is highest weight of weight d , then the composition factors of $\mathcal{F}^{(n)}L$ are either $f^n L$ or in $\mathcal{C}(< d)$.

Proof.

The simples in $\mathcal{C}(< d)$ of weight $d - 2n$ are distinguished by the fact that \mathcal{E}^n kills them. Thus, we need only note that $f^n L$ appears with composition multiplicity $\binom{d}{n}$, and that

$$\mathcal{E}^{(n)}f^n L \cong L \quad \mathcal{E}^{(n)}\mathcal{F}^{(n)}L \cong L^{\oplus \binom{d}{n}}.$$

This is only possible if all other composition factors are in $\mathcal{C}(< d)$. □

Kashiwara operators

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Lemma

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Corollary

The subcategories $\mathcal{C}(< d)$ have induced \mathfrak{sl}_2 -actions. There is a fully faithful functor commuting with \mathcal{E} and \mathcal{F} of the form

$$\iota_L: \mathcal{A}_{d-2n}^d \rightarrow \mathcal{C}_{d-2n}(\leq d) / \mathcal{C}_{d-2n}(< d) \quad \iota(M) = M \otimes_{H_n} \mathcal{F}^{(n)}L.$$

Checking isomorphisms

Theorem

If a 2-morphism $p: F \rightarrow G$ in \mathfrak{A} induces an isomorphism between functors in \mathcal{A}^d for every d , then it induces an isomorphism for every integrable categorical \mathfrak{sl}_2 action on an artinian abelian category.

Proof.

Assume not.

- Then let d be the minimal integer such that p fails to induce an isomorphism on $\mathcal{C}(\leq d)$. In this case, the natural transformation induced by p on functors on $\mathcal{C}(\leq d)/\mathcal{C}(< d)$ is not an isomorphism.
- Since the functors E and F are exact, we can assume that the module M on which it fails is a simple.
- Thus M is in the image of the functor ι_L for some L . By assumption, p induces an isomorphism on this subcategory. This is a contradiction. \square

Applications

The first application is that we can tighten up the relations in our 2-category:

Definition

If $n \geq 0$, let $\rho_n: \mathcal{F}\mathcal{E} \oplus \text{id}^{\oplus n} \rightarrow \mathcal{E}\mathcal{F}$ be the sum of the maps

$$(1_{\mathcal{E}} \otimes 1_{\mathcal{F}} \otimes \epsilon)(1_{\mathcal{E}} \otimes \psi \otimes 1_{\mathcal{E}})(\iota \otimes 1_{\mathcal{F}} \otimes 1_{\mathcal{E}}), \iota, (y \otimes 1_{\mathcal{E}})\iota, (y^2 \otimes 1_{\mathcal{E}})\iota, \dots, (y^{-n-1} \otimes 1_{\mathcal{E}})\iota,$$

If $n \leq 0$, let $\rho_n: \mathcal{F}\mathcal{E} \rightarrow \mathcal{E}\mathcal{F} \oplus \text{id}^{\oplus -n}$ be the sum of the maps

$$(1_{\mathcal{E}} \otimes 1_{\mathcal{F}} \otimes \epsilon)(1_{\mathcal{E}} \otimes \psi \otimes 1_{\mathcal{E}})(\iota \otimes 1_{\mathcal{F}} \otimes 1_{\mathcal{E}}), \epsilon, \epsilon(1_{\mathcal{E}} \otimes y), \epsilon(1_{\mathcal{E}} \otimes y^2), \dots, \epsilon(1_{\mathcal{E}} \otimes y^{n-1})$$

Theorem

In any integrable categorical \mathfrak{sl}_2 action on an artinian abelian category, the map ρ_n is an isomorphism on objects of weight n .

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Theorem

In any integrable categorical \mathfrak{sl}_2 action on an artinian abelian category, the map ρ_n is an isomorphism on objects of weight n .

Applications

Of course, the proof is to check it on \mathcal{A}^d .

In principle, this is just a problem of linear algebra; we already know the two sides have the same dimension, so we just have to check surjectivity. Chuang and Rouquier's approach is essentially to do just that.

It's instructive to look at the case where M is a highest weight object. In this case, the map $\rho_n(M): M^{\oplus n} \rightarrow \mathcal{EFM}$ is just given by the unit of the adjunction times $1, y, \dots, y^{n-1}$.

Taking $\text{Hom}(M, -)$, this is the claim that $\text{Hom}(M, \mathcal{EFM})$ is a free module of rank n on $\text{End}(M)$, generated by these elements.

That is to say, $\text{Hom}(\mathcal{FM}, \mathcal{FM})$ is spanned as a left module over $\text{End}(M)$ by $1, \dots, y^{n-1}$; this is a fact we already knew.

Applications

There is another approach:

Lauda's solution

On \mathcal{A}^d , the functors \mathcal{E} and \mathcal{F} aren't just adjoint, they're biadjoint (by Poincaré duality). Thus, there are also diagrams:



Lauda uses these to simply write a formula for the inverse to this map:

$$\begin{array}{c} \downarrow \\ \uparrow \end{array} = - \begin{array}{c} \nearrow \\ \searrow \end{array} + \sum_{a+b+c=-1} \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ \curvearrowright \end{array}$$

Another 2-category

This allows us make the $[E, F]$ relation genuinely categorical: say two objects are isomorphic is not a relation in a 2-category, saying that a particular map between them is an isomorphism is.

Definition

Let \mathfrak{A} be the category \mathfrak{A} with a formal inverse ρ_n^{-1} added to its set of 2-morphisms. This acts on any integrable categorical \mathfrak{sl}_2 action.

We can also make an even “tighter” 2-category by adding an adjunction in the other direction between \mathcal{E} and \mathcal{F} , and adding in some extra relations (principally the one on the page before). However, it’s not clear that this will act on every categorical \mathfrak{sl}_2 action as \mathfrak{A} does.

Biadjunction



Another important consequence of this theorem is to also prove that \mathcal{E} and \mathcal{F} must be biadjoint.

It's worth noting that while an adjunction one way is unique up to unique isomorphism, biadjunctions are not unique. There are genuinely different choices one can make.

Implicit in the relation of Lauda which I put up is one choice:

Definition

Let ι' be

- if $n < 0$, the component of ρ_n^{-1} matching 
- if $n \geq 0$, then it is  acting on the component of ρ_n^{-1} matching the crossing.

Biadjunction

On the minimal categorification \mathcal{A}^n , this is just the biadjunction coming from Poincaré duality; in particular, it induces an isomorphism between \mathcal{F} and the right adjoint of \mathcal{E} .

Theorem (Rouquier)

Thus, the same is true for any integrable categorical \mathfrak{sl}_2 -action on a \mathbb{k} -linear category.

Reflection isomorphisms

The final application of this theorem is the most impressive: the construction of derived equivalences between weight spaces.

We know that in an integrable representation V of \mathfrak{sl}_2 , there is an action of the group SL_2 . In particular, the group element $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ induces an isomorphism $s: V \rightarrow V$ with the following properties:

$$sHs^{-1} = -H \quad sEs^{-1} = -F \quad sFs^{-1} = -E$$

In particular, $s: V_n \rightarrow V_{-n}$ is an isomorphism between opposite weight spaces.

Thus, this should categorify to an equivalence of categories $\mathcal{C}_n \cong \mathcal{C}_{-n}$ between the weight spaces.

Unfortunately, this is obviously impossible. That -1 is the first issue, but it's much worse than that.

Derived equivalences

In fact, there are lots of \mathfrak{sl}_2 actions where s sends the class of a module in $V_n = K^0(\mathcal{C}_n)$ to something that can't be a class of a module, or -1 times a class of a module.

In fact, there are lots of examples where $\mathcal{C}_n \not\cong \mathcal{C}_{-n}$ as an abelian category!

Well, that doesn't bode well.

Derived equivalences

Luckily, there's a fix for this:

They might not be equivalent as abelian categories, but their derived categories are equivalent.

For those of you who don't like derived categories, don't worry too much. You can think instead of the category $\mathcal{K}^b(\mathcal{C}_n)$ of bounded (or bounded above/below) complexes of objects in \mathcal{C}_n , with morphisms considered up to homotopy.

Of course, every functor $\mathcal{C}_n \rightarrow \mathcal{C}_{-n}$ induces a functor $\mathcal{K}^b(\mathcal{C}_n) \rightarrow \mathcal{K}^b(\mathcal{C}_{-n})$, but there are many, many more.

Derived equivalences

For example, consider the homotopy category of modules over $A = \mathbb{C}[t]/(t^2)$.

There's a non-trivial exact autoequivalence that sends

$$\begin{array}{ccccccc} A & & A & & \mathbb{C} & & A & & A \\ \downarrow & \mapsto & \downarrow & & \downarrow & \mapsto & \downarrow & \mapsto & \downarrow \\ 0 & & 0 & & 0 & & \mathbb{C} & & A \\ & & & & & & & & \downarrow \\ & & & & & & & & \mathbb{C} \end{array}$$

Though this preserves homology, it doesn't preserve homotopy classes!

The action of s

There's a formula for how s acts on an element of weight n in a \mathfrak{sl}_2 module:

$$sv = \begin{cases} F^{(n)}v - F^{(n+1)}Ev + F^{(n+2)}E^{(2)}v - \dots & n \geq 0 \\ (-1)^n (E^{(n)}v - FE^{(n+1)}v + F^{(2)}E^{(n+2)}v - \dots) & n \leq 0 \end{cases}$$

When we see an alternating formula like this, it looks like an Euler characteristic and we can try to categorify it with complexes.

Let $\Theta^i = \mathcal{F}^{(n+i)}\mathcal{E}^{(i)}$ if $i, n+i \geq 0$, and 0 otherwise. We will try to build a complex out of these pieces, which we then expect will categorify s .

The action of s

If this were $\mathcal{F}^{n+i}\mathcal{E}^i$, then I would have a very good guess about what to do: just use the counit $\epsilon: \mathcal{F}\mathcal{E} \rightarrow \text{id}$ on the middle pair of functors, and get an induced map $\mathcal{F}^{n+i}\mathcal{E}^i \rightarrow \mathcal{F}^{n+i-1}\mathcal{E}^{i-1}$.

Of course, $\mathcal{F}^{(n)}$ is a summand of \mathcal{F}^n , so one can just use the inclusion and projection. One just has to be a bit careful, since it's actually a summand in a bunch of different ways. We have an isomorphism

$$\mathcal{F}^{(n)}\mathcal{F} \cong \mathcal{F}^{(n+1)} \otimes H^*(\mathbb{P}^1)$$

and we should use the unit and counit of this Frobenius algebra to define the inclusion and projection $\mathcal{F}^{(n+1)} \xrightarrow{u} \mathcal{F}^{(n)}\mathcal{F} \xrightarrow{c} \mathcal{F}^{(n)}\mathcal{F}$ (in particular, these are not splittings of a single inclusion).

Let $d^i: \Theta^i \rightarrow \Theta^{i-1}$ be the map

$$\mathcal{F}^{(n+i)}\mathcal{E}^i \xrightarrow{u \otimes c^*} \mathcal{F}^{(n+i-1)}\mathcal{F}\mathcal{E}\mathcal{E}^{(i-1)} \xrightarrow{\epsilon} \mathcal{F}^{(n+i-1)}\mathcal{E}^{(i-1)}$$

The action of s

Definition

Let Θ denote the complex $\cdots \rightarrow \Theta^i \rightarrow \Theta^{i-1} \rightarrow \cdots$.

If we apply this complex of functors to a complex of modules, we get a double complex, which we collapse down to a single one.

Theorem (Chuang-Rouquier, Cautis-Kamnitzer-Licata)

This complex of functors defines an equivalence of categories
 $K^b(\mathcal{C}_n) \cong K^b(\mathcal{C}_{-n})$.

Proof.

Of course, we just need to check this for \mathcal{A}^d for every d . Let $k = (d - n)/2$. On the projective module H_k itself. In fact, in this case, we can check that actually, $\Theta(H_k) \cong H_{d-k}[k]$ □

The complex Θ

I didn't find a proof of this I was really happy with, so let me try to give a heuristic argument.

You should think of Θ as corresponding to the locus:

$$N = \{(V, V') \in \mathbf{Gr}(d-k, \mathbb{C}^d) \times \mathbf{Gr}(k, \mathbb{C}^d) \mid V + V' = \mathbb{C}^d\}$$

Note that $H^*(\mathbf{Gr}(k, \mathbb{C}^d)) \cong H_c^*(N) \cong H^*(\mathbf{Gr}(d-k, \mathbb{C}^d))$.

The last non-zero term of $\Theta(H_k)$ is $\mathcal{F}^{(n+k)}\mathcal{E}^{(k)}$, which corresponds to $H^*(\mathbf{Gr}(d-k, \mathbb{C}^d) \times \mathbf{Gr}(k, \mathbb{C}^d))$, and the homotopy equivalence we want is induced by the map $H_c^*(N) \rightarrow H^*(\mathbf{Gr}(d-k, \mathbb{C}^d) \times \mathbf{Gr}(k, \mathbb{C}^d))$.

We can analyze the kernel of this map using topology: it's $H^*(N^c)$. This has a filtration according to the dimension of the intersection, which you can think of as corresponding to the other terms.

A few moments with perverse sheaves

Actually, let me be fancy for a moment. What we want to do is analyze the perverse sheaf $i_*\mathbb{C}_N$. This has a weight filtration that breaks it into semi-simple pieces.

Point counting on Grassmannians shows that the m th layer of weight filtration is

$$(p_m)_*\mathbb{C}_{\mathrm{Gr}(d-k \supset m \subset k)} \quad p_m: \mathrm{Gr}(d-k \supset m \subset k) \rightarrow \mathrm{Gr}(d-k, \mathbb{C}^d) \times \mathrm{Gr}(k, \mathbb{C}^d).$$

The differentials in the complex correspond to extensions between these sheaves that glue them together to get $i_*\mathbb{C}_N$.

Thus, all of the content of the equivalence is just that $H_c^*(N) \otimes_{H_k} -$ is an equivalence.

Examples

However, on categorifications that glue together different simples, life isn't nearly so easy. For example, let $d = 2$ and consider all modules over H_k . The equivalence between modules over $H_0 \cong \mathbb{C} \cong H_2$ isn't very interesting, but the autoequivalence between modules over H_1 is.

In this case, Θ corresponds to the complex $\mathcal{F}\mathcal{E} \rightarrow \text{id}$. For the modules H_1 and \mathbb{C} , this goes to

$$\Theta(H_1) = H_1 \otimes_{\mathbb{C}} H_1 \rightarrow H_1 \quad \Theta(\mathbb{C}) = H_1 \rightarrow \mathbb{C}$$

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However, on categorifications that glue together different simples, life isn't nearly so easy. For example, let $d = 2$ and consider all modules over H_k . The equivalence between modules over $H_0 \cong \mathbb{C} \cong H_2$ isn't very interesting, but the autoequivalence between modules over H_1 is.

In this case, Θ corresponds to the complex $\mathcal{F}\mathcal{E} \rightarrow \text{id}$. For the modules H_1 and \mathbb{C} , this goes to

$$\Theta(H_1) = H_1[1] \quad \Theta(\mathbb{C}) = H_1 \rightarrow \mathbb{C}$$

This shows that Θ is (up to shift) the derived equivalence we considered before!

Symmetric groups

The most important application of this is basically the only other categorification we know, the representations of $\mathbb{k}[S_n]$ -mod.

Theorem (Chuang-Rouquier; “Broué’s conjecture”)

If a block of $\mathbb{k}[S_n]$ and a block of $\mathbb{k}[S_m]$ have the same decomposition group, then they are derived equivalent.

OK, I didn’t explain several of these words. Remember that simples are assigned a weight according to the contents of the boxes in their Young diagram. The simples that have the same weight for \mathcal{E}_a and \mathcal{F}_a form a **block**; they’re a group that can stick to each other in extensions. Thus, the functor Θ_a for the \mathfrak{sl}_2 action $(\mathcal{E}_a, \mathcal{F}_a)$ sends one block to another.

Symmetric groups

Maybe the most down to earth way to think about this is that every block of S_n is derived equivalent to the block of the trivial module in S_{mp} where $p = \text{char}(\mathbb{k})$ (this is the case where each elements of \mathbb{F}_p show up m times in the diagram).

In any other case, there is some $i \in \mathbb{k}$ which appears the largest number of times, and Θ_i will give a derived equivalence to a block with fewer boxes.

Thus, as complicated as the representation theory of symmetric groups over finite fields can be, there's a precise sense in which the complicatedness is contained in these special blocks.

Next time...

Next time, we'll talk about how to categorify tensor products of representations of \mathfrak{sl}_2 , and what that might have to do with invariants of knots.