Higher representation theory in algebra and geometry: Lecture III

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UVA

February 20, 2014

References

For this lecture, useful references include:

- Webster, *Tensor product algebras, Grassmannians and Khovanov homology* (basically covers all this material, though in a slightly different presentation)
- Bar-Natan, *Khovanov's homology for tangles and cobordisms* (introduces $2-TL$ in a slightly more general form, and covers original definition of Khovanov homology)
- Khovanov, *A categorification of the Jones polynomial* and *A functor-valued invariant of tangles* (more detailed discussion of underlying topology)

The slides for the talk are on my webpage at:

`http://people.virginia.edu/~btw4e/lecture-4.pdf`

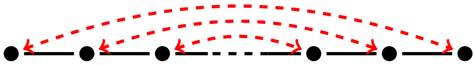
You can also find some proofs that I didn't feel like going through in class at:

`https://pages.shanti.virginia.edu/Higher_Rep_Theory/`

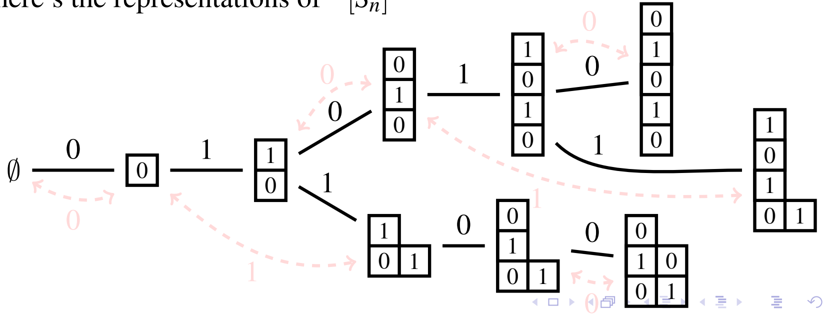
Reminder about last time

We've now spent a lot of time talking about the general structure of categorical $\mathfrak{sl}_2$ actions (maybe too much?), but have a relatively thin store of examples:

- the category $\mathcal{A}^d$ of modules over $H^*(\mathsf{Gr}(k, \mathbb{C}^d))$ and



- there's the representations of $[S_n]$



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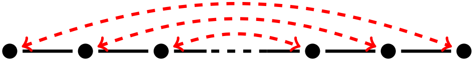

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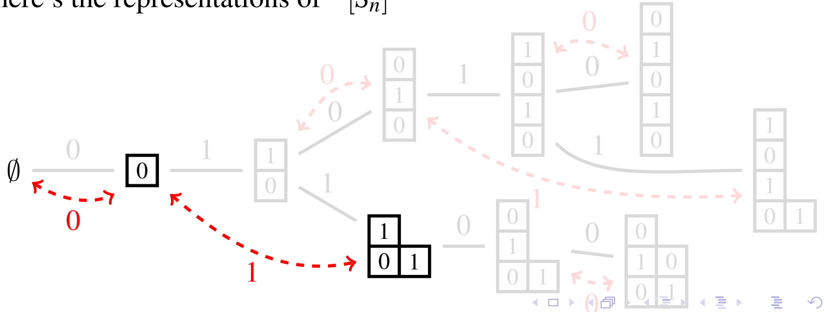

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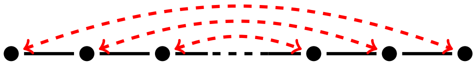

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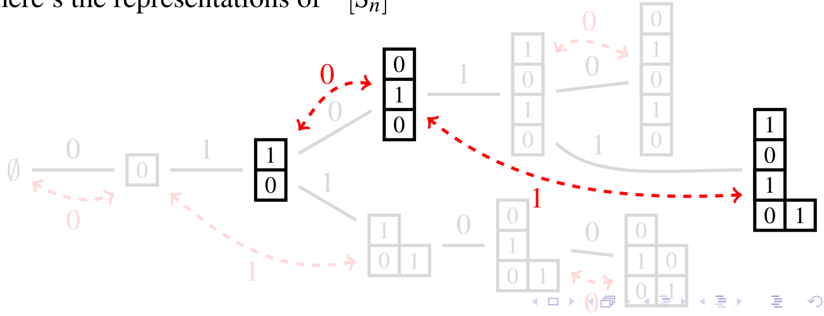

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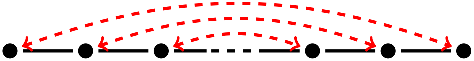

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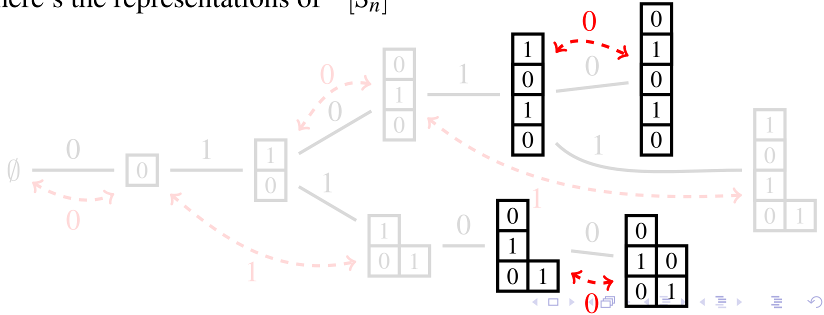


More categorical actions?

So, how are we going to find more? Well, it turns out that if we look at all modules over $H^*(\mathsf{Gr}(k, \mathbb{C}^d))$ as an *additive* category, there's a lot of interesting stuff in there. In fact, so much interesting stuff that it's unwise to just start poking around.

Instead, let's think a bit more about the geometry of Grassmannians. The simple module $\mathrm{Sym}^d(\mathbb{C}^2)$ appears geometrically as the space of constant function on Grassmannians under the correspondence picture.

And, indeed, we categorified it as projective modules over $H^*(\mathsf{Gr}(k, \mathbb{C}^d))$ which correspond to the sheaf of constant functions in some vector space V on $\mathsf{Gr}(k, \mathbb{C}^d)$, since

$$\mathbb{H}^*(\mathsf{Gr}(k, \mathbb{C}^d);\ \ {}_{\mathsf{Gr}} \otimes V) \cong V \otimes H^*(\mathsf{Gr}(k, \mathbb{C}^d)).$$

More categorical actions?

Thus, if we start with a more interesting partition $\mathsf{Gr}(k, \mathbb{C}^d) = \dot{\sqcup} X_\alpha$ on the Grassmannian, we should get a more interesting categorification from it:

The sheaves which are constant when restricted to each X_α should categorify functions which are constant on each X_α.

Of course, there's one partition on the Grassmannian we all know and love: the Schubert cells.

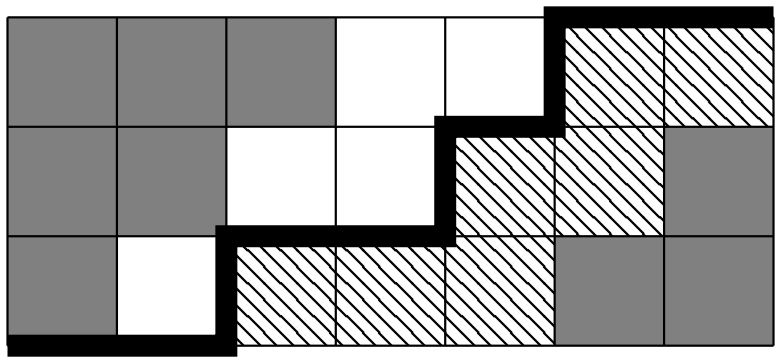
Schubert cells

So, pick your favorite flag $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^d$. Let $\kappa\colon [1,d] \to [0,k]$ be a weakly increasing function, and let

$$X_\kappa = \{V \in \mathsf{Gr}(k, \mathbb{C}^d) | \dim(V \cap \mathbb{C}^{m-1}) = \kappa(m)\}$$

Typically, people don't want to take an arbitrary κ here, but one that is **Schubert**, i.e. which satisfies $\kappa(1) = 0$ and $\kappa(m+1) \leq \kappa(m) + 1$.

These conditions say that when we draw the graph of κ, we stay inside a lozenge, and when the rows are "straightened," we end up with the diagram of a partition under the graph (in this case, $(3,2,2)$).



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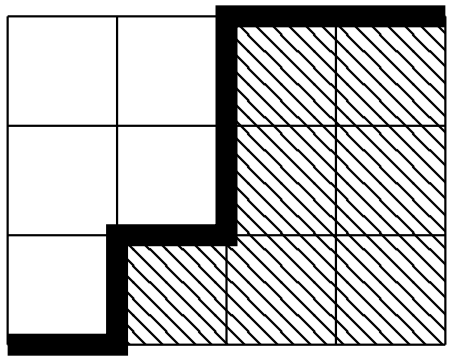


Schubert varieties

We can also associate Schubert varieties to subsets of $[1,d]$. Let

$$I_\kappa = \{m | \kappa(m+1) > \kappa(m)\};$$

More geometrically, if $V \in X_\kappa$, then $m \in I_\kappa$ if and only if $V \cap \mathbb{C}^m \not\subset \mathbb{C}^{m-1}$.

If you think of $2^{[1,d]}$ as a "Grassmannian over $\mathbb{F}_1$," then this is just saying every Schubert cell has exactly one point.

For any subset $I \subset [1,d]$, let $v_I = v_1 \otimes \cdots v_d \in (\mathbb{C}^2)^{\otimes d}$, where

$$v_i = \begin{cases} (1,0) & i \notin I \\ (0,1) & i \in I \end{cases}$$

These obviously give a basis of $(\mathbb{C}^2)^{\otimes d}$.

Schubert varieties

Theorem

The functions on $\mathsf{Gr}(k, \mathbb{F}_p^d)$ *which are constant on Schubert varieties are isomorphic to* $(\mathbb{C}^2)^{\otimes d}$ *as a representation of the quantum group* $U_{\sqrt{p}}(\mathfrak{sl}_2)$ *with*

$$v_{I_\kappa} \mapsto p^{-1/2 \dim X_\kappa} \chi_{X_\kappa}.$$

I hate defining $U_q(\mathfrak{sl}_2)$, but I should say something about what $\otimes$ means here.

Instead of acting on a tensor product like a Lie algebra would, the generators E and F in $U_{\sqrt{p}}(\mathfrak{sl}_2)$ act by

$$E(v \otimes w) = Ev \otimes w + q^{\mathrm{wt}(v)} v \otimes Ew \qquad F(v \otimes w) = q^{-\mathrm{wt}(w)} Fv \otimes w + v \otimes Fw.$$

The most important thing here is that switching tensor factors in no longer a map of representations.

Schubert varieties

Thus, we expect that looking at sheaves which are constant on the Schubert cells will give us a categorification of the tensor product.

Now, I'm sure that some of you will say "Ah, yes, of course, perverse sheaves (or D-modules) smooth on the Schubert cells. What a reasonable thing to think about." But if you don't, don't worry. There's a lower tech way of thinking about these.

Schubert varieties

To do this, we need to use the Schubert variety

$$\bar{X}_\kappa = \{V \in \mathsf{Gr}(k, \mathbb{C}^d) | \dim(V \cap \mathbb{C}^{m-1}) \geq \kappa(m)\}$$

and its natural resolution of singularities

$$\tilde{X}_\kappa = \{V_1 = \{0\} \subset V_1 \subset \cdots \subset V_\ell \in \mathsf{Gr}(k, \mathbb{C}^d) | \dim V_m = \kappa(m), V_m \subset \mathbb{C}^{m-1}\}$$

This resolution is a way of "getting at" the Schubert smooth perverse sheaves, by looking at the pushforward of the constant sheaf by the map $p_\kappa : \tilde{X}_\kappa \to \bar{X}_\kappa$.

More generally, for *any* κ, we'll want to look at

$$Y_\kappa = \{W_1 \subset \cdots \subset W_k | \dim W_j = j, W_{\kappa(m)} \subset \mathbb{C}^{m-1}\}$$

This is actually a smooth Schubert variety in the full flag variety.

Convolution in cohomology

Let $\tilde{X} = \sqcup_\kappa \tilde{X}_\kappa$ and $X = \sqcup_k \mathsf{Gr}(k, \mathbb{C}^d)$. We have a map $p\colon \tilde{X} \to X$ and interested in the category of summands of $p_* \quad_{\tilde{X}}$ (which is the same as semi-simple Schubert smooth perverse sheaves).

Proposition

Chriss and Ginzburg tell us that we can understand this in terms of algebraic topology:

$$\mathrm{Ext}^\bullet(p_* \quad_{\tilde{X}}, p_* \quad_{\tilde{X}}) \cong H_*(\tilde{X} \times_X \tilde{X})$$

intertwing the Yoneda and convolution products.

So, if you want to be fancy, you can think of the LHS as the basic object and the RHS as just some concrete way of getting at it. Alternatively, you can think of RHS as the basic object that interests us, and the LHS as some obscure motivation you don't need to worry about.

Actually, $p_* \quad_X$ is a bad choice. It's better to consider $\pi\colon Y = \sqcup_\kappa Y_\kappa \to X$, and the Morita equivalent algebra $T^d := \mathrm{Ext}^\bullet(\pi_* \quad_Y, \pi_* \quad_Y) \cong H_*(Y \times_X Y)$.

Convolution in cohomology

OK, that may not seem like an improvement, but it is, since it allows us to think concretely. For example, we have an algebra map $H_*(Y) \to T^d$ induced by the diagonal inclusion $Y \hookrightarrow Y \times_X Y$. Inside $H^*(Y)$, we have

- the fundamental classes of the different components $e_\kappa = [Y_\kappa]$.
- We let $y_i e_\kappa = c_1(\mathcal{L}_i) \cap [Y_\kappa]$ where $\mathcal{L}_i = W_i/W_{i-1}$ is a tautological line bundle on Y_κ.

The other generators come from

$$Z_{i,\kappa,\kappa'} = \{(W_\bullet, W'_\bullet) \in Y_\kappa \times Y_{\kappa'} \mid W_k = W'_k \text{ for all } k \neq i\};$$

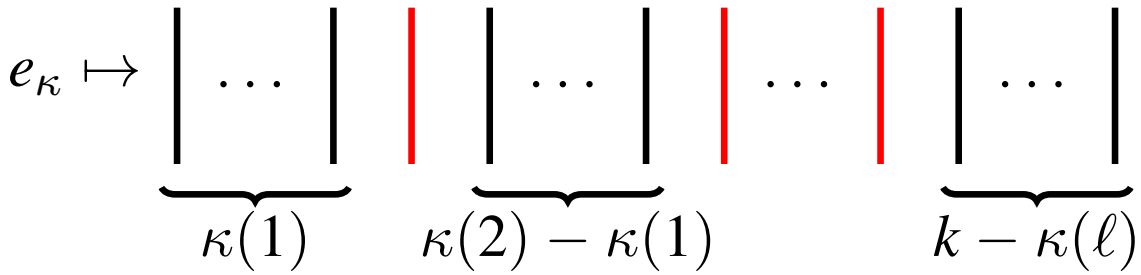
endowed with projection maps $h_1, h_2 \colon Z_i \to Y$. I allow $i = 0$.

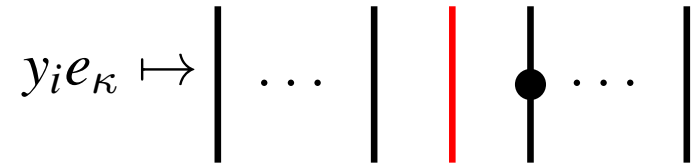
Proposition

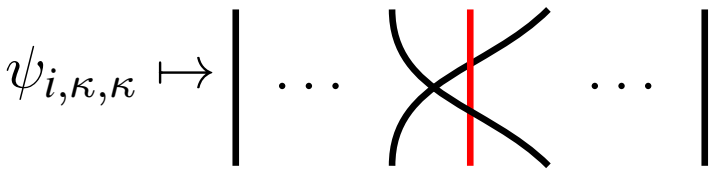
The algebra T^d is generated by e_κ, y_i and $\psi_{i,\kappa,\kappa'} = (h_1 \times h_2)_*[Z_{i,\kappa,\kappa'}]$.

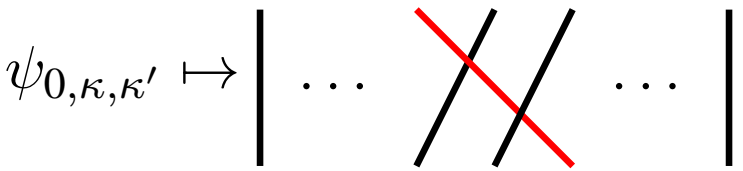
Diagrammatics

These elements can be described using diagrams.









Relations

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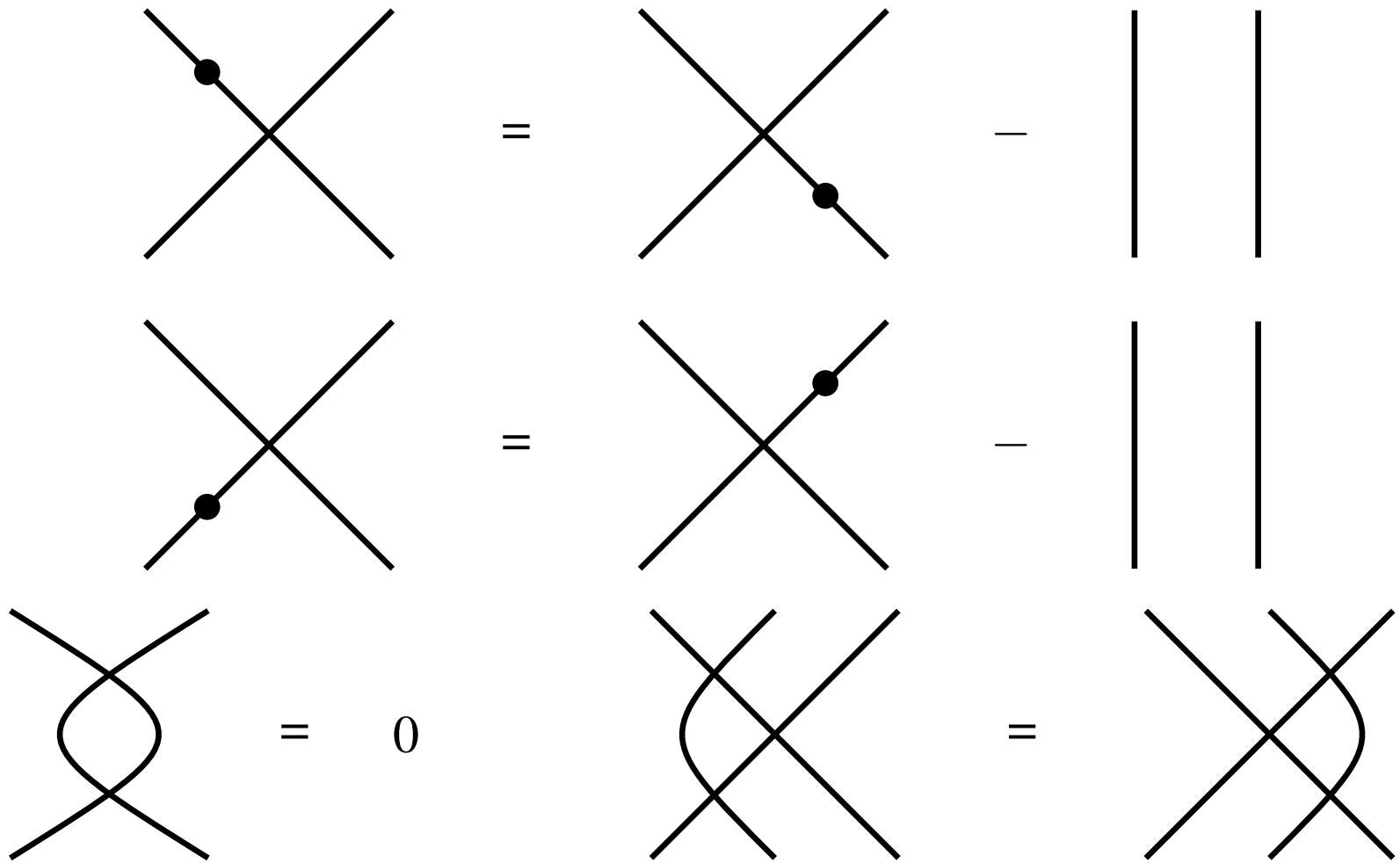

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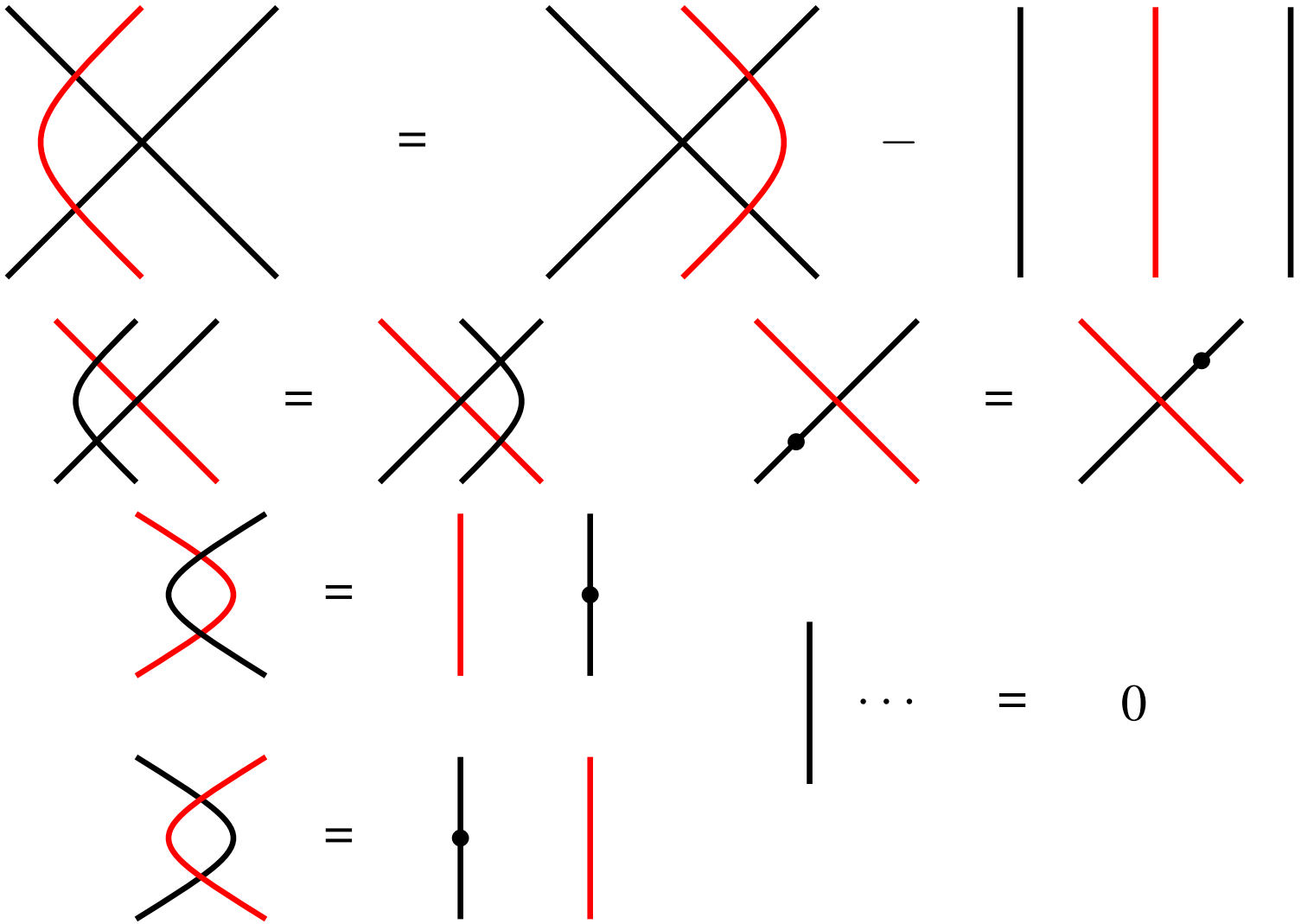


Tensor product algebras

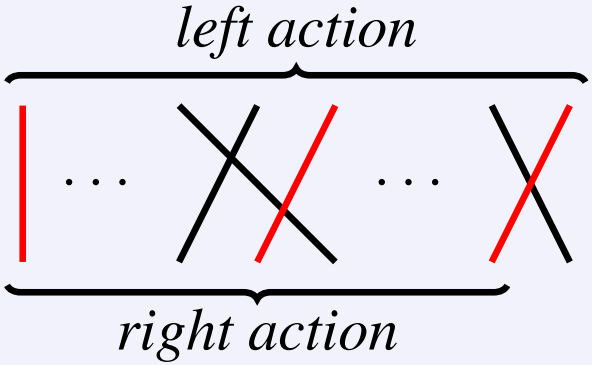
Proposition

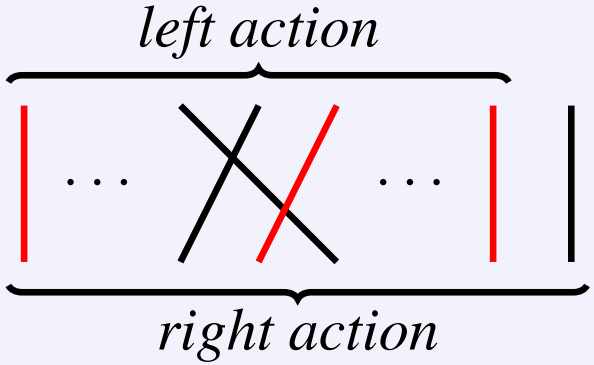
The algebra T^d has a purely diagramatic presentation as linear combinations of such diagrams (we call them **Stendhal diagrams***) modulo the local relations on the previous page.*

There are a number of results that have both algebraic and geometric proofs.

Theorem

The category $\mathfrak{V}^d = T^d$-mod carries a categorical $\mathfrak{sl}_2$-action via tensor product with the bimodules

$\mathfrak{F} =$ 

$\mathfrak{E} =$ 

Grothendieck group

Theorem

The Grothendieck group $K(T^d) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to $(\mathbb{C}^2)^{\otimes d}$ as a $\mathfrak{sl}_2$-module.

Unfortunately, using our current technology, this isomorphism looks a bit strange. The class of the projective module $P_\kappa = T^d e_\kappa$ goes to the function on $X(\mathbb{F}_p)$ given by

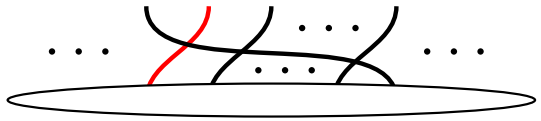
$$\tau_\kappa(V) = \#\{W_1 \subset \cdots \subset W_k \in Y_\kappa \mid W_k = V\}$$

In algebraic terms, if $\kappa(1) = 0$, this is

$$[P_\kappa] \mapsto p_\kappa := F^{k-\kappa(\ell)}(F^{\kappa(\ell)-\kappa(\ell-1)}(\cdots F^{\kappa(2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Representations

There are also **standard modules** S_κ, the quotient of P_κ by the relations



Under the isomorphism $K(T^\ell\text{-mod}) \cong (\mathbb{C}^2)^{\otimes \ell}$, if κ is Schubert, then

$$[S_\kappa] \to F^{\kappa(2)-\kappa(1)}v_+ \otimes \cdots \otimes F^{k-\kappa(\ell)}v_+$$

Proposition

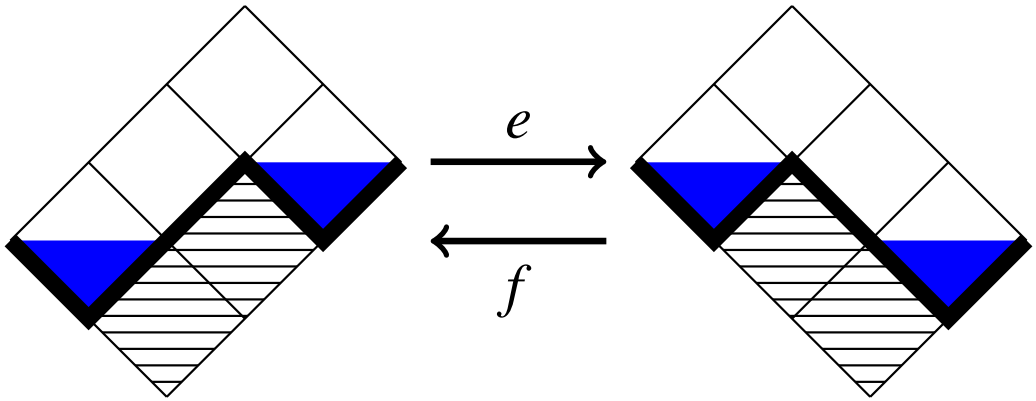
Every simple T^d-module is the quotient of a unique standard module for κ Schubert.

The crystal structure

The crystal structure on simples has a nice description.

1. Rotate your Young diagram to "Russian notation."
2. Imagine pouring water on it, so that it has some "lakes" where water pools, and "land" which is left exposed. Note that the land slopes upward until it reaches some peak, and then turns downward.
3. the operator f turns the leftmost downward sloping patch of land upward, leaving all the lakes alone. The operator e turns the rightmost upward sloping patch downward.

Note that this operation depends on the box that the diagram is in, not just the partition.



Grading

I've mostly not been bothering with the gradings on things, but they are all around you. In particular, T^d is naturally a graded algebra, but not with the obvious grading. Instead, we have to use the rule

$$\deg \times = -2 \qquad \deg \times = 1 \qquad \deg \times = 1 \qquad \deg \underset{i}{\bullet} = 2$$

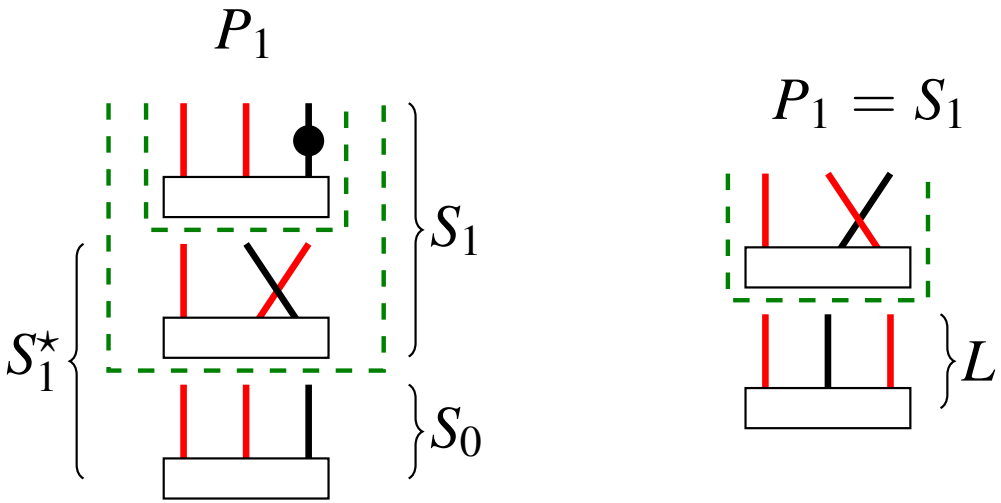
Note that this rule makes the relations homogeneous.

This also means that the functors $\mathcal{E}$ and $\mathcal{F}$ of the $\mathfrak{sl}_2$ can be upgraded to functors between categories of graded modules such that the natural transformations of $\mathfrak{A}$ are homogeneous.

Keeping track of grading shift quantizes everything. The **graded Grothendieck group** $K_q(T^d\text{-gmod}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a $\mathbb{C}[q, q^{-1}]$ module, isomorphic to $V_1^{\otimes d} = (\mathbb{C}^2[q, q^{-1}])^{\otimes d}$ as a module over $U_q(\mathfrak{sl}_2)$.

Representations

Let's look at the structure of the case of T^2 with one black line. We'll denote things by the value of $\kappa(1)$.



$$[P_0] = v_+ \otimes v_- + q^{-1} v_- \otimes v_+ \quad [P_1 \cong S_1] = v_- \otimes v_+ \quad [S_0] = v_+ \otimes v_-$$

$$[S_1^\star] = v_- \otimes v_+ + (q - q^{-1}) v_+ \otimes v_- \quad [L] = v_- \otimes v_+ - q^{-1} v_+ \otimes v_-$$

The only module we can see in this diagram killed by $\mathcal{E}$ is L.

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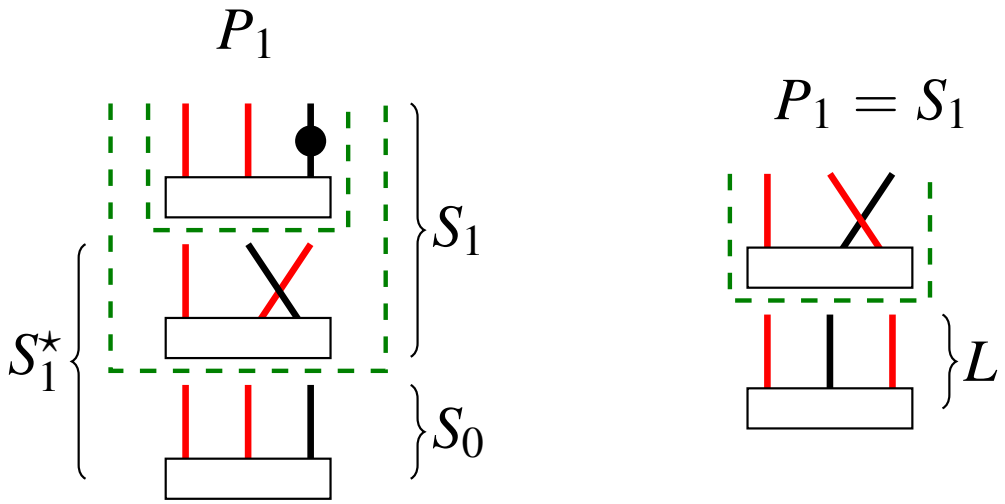

$$\dim_q([\mathcal{E}P_0]) = q + q^{-1} \quad \dim_q([\mathcal{E}P_1]) = 1 \quad \dim_q([\mathcal{E}S_0]) = q$$
$$\dim_q([\mathcal{E}S_1^\star]) = q^{-2} \quad \dim_q([\mathcal{E}L]) = 0$$

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Double centralizer

There's one very special projective P_n over T_n^d. When $d = n$, we have $P_d \cong T_d^d \cong \quad$, and otherwise $P_{d-2k} \cong \mathcal{F}^{(k)} P_n$.

Theorem

We have an isomorphism $\mathrm{End}(P_{d-2k}) \cong H_k^d := H^*(\mathsf{Gr}(k, \mathbb{C}^d); \quad)$, *and the functor*

$$\mathbb{V} := \mathrm{Hom}(P_{d-2k}, -) \colon T_{d-2k}^d\text{-mod} \to H_k\text{-mod}$$

is **fully faithful on projectives**. *That is,* $\mathbb{V}$ *defines an equivalence of* T_{d-2k}^d-proj *to a subcategory of* H_k-mod.

We have that $\mathbb{V} T^d e_\kappa = H^*(Y_\kappa; \quad)$.

In fact, $\mathrm{im}\,\mathbb{V}$ is the smallest additive idempotent complete category closed under $\mathcal{E}$, $\mathcal{F}$, and pullback by the map $H_k^d \to H_k^{d-1}$ given by the embedding $\mathbb{C}^{d-1} \to \mathbb{C}^d$.

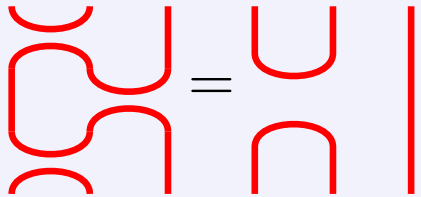
Why tensor?

Tensor products play an important role in the theory of quantum groups, but for me, they're especially important because of connections to topology.

Definition

The **Temperley-Lieb** *algebra TL_n is generated elements b_i for $i = 1, \ldots, n-1$ subject to the relations*

$$b_i^2 = q + q^{-1} b_i \qquad b_i b_{i\pm 1} b_i = b_i$$

$b_i =$ $= q + q^{-1}$ 

Proposition

The algebra TL_d acts faithfully on the tensor product $V_1^{\otimes d}$; on $V_1 \otimes V_1$, the endomorphism b is $q + q^{-1}$ times projection to invariants. After tensoring with to $\mathbb{C}(q)$, this is the full endomorphism algebra

The Temperley-Lieb category

Actually, we can think of all of these together as a category TL.

- The objects of the category are positive integers $\mathbb{Z}_{\geq 0}$.
- The morphisms $n \to m$ are $\mathbb{C}(q)$-linear combinations of crossingless planar diagrams in $\mathbb{R} \times [0,1]$ that meet $y = 0$ in n points and $y = 1$ in m (up to isotopy).
- composition is stacking, with every circle becoming multiplication by $q + q^{-1}$.

We have a functor from TL to $U_q(\mathfrak{sl}_2)$ -mod sending $d \mapsto V_1^{\otimes d}$.

Jones polynomials

Definition

Recall that a (n, m) **tangle** *is a non-intersecting collection of smooth curves in* $\mathbb{R}^2 \times [0, 1]$ *which meets* $z = 0$ *at* $(0, 1, 0), \ldots, (0, n, 0)$ *and* $z = 1$ *at* $(0, 1, 1), \ldots, (0, m, 1)$ *(as usual, considered up to isotopy).*

A **ribbon tangle** *is also equipped with a framing, parallel to the y-axis at* $z = 0, 1$.

We can think of (ribbon) tangles as morphisms in a category tang whose objects are the integers $\mathbb{Z}_{\geq 0}$.

Theorem (Jones)

There's a functor tang $\to$ *TL which sends*

Categorified Jones polynomials

One obvious question is whether one can categorify this story. Luckily, the answer is yes!

So, what would that mean?

- Every vector space should turn into a category $V_i^{\otimes d} \mapsto T^d$ -mod.
- Every morphism in the Temperley-Lieb category should turn into a functor, as should the maps we assigned to tangles.

This is where we become glad that we invested in turning everything into algebras. If you really want we can discuss how to do this all using Whittaker sheaves, for the moment, I'm glad not to. Instead I just have to write down some bimodules.

The case of $\mathfrak{sl}_2$

The cup (up to shift) is associated to *derived* tensor product with the bimodule:

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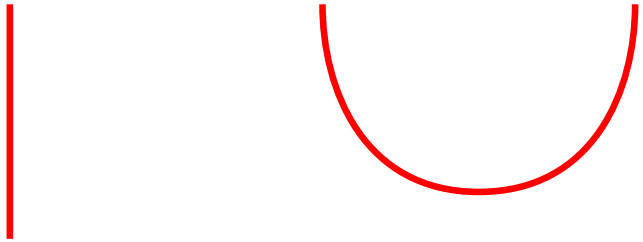

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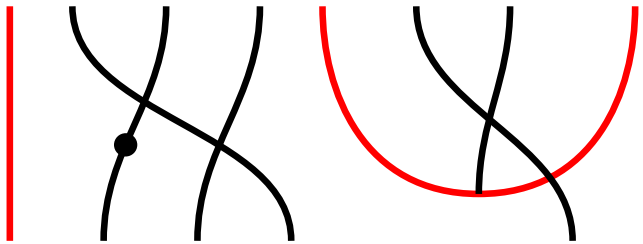

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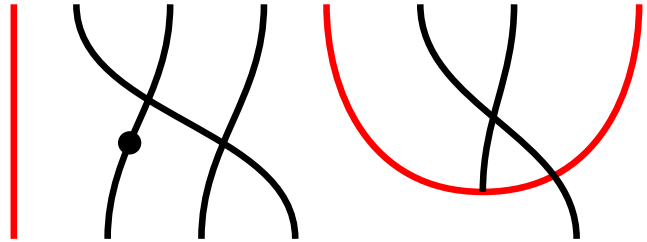
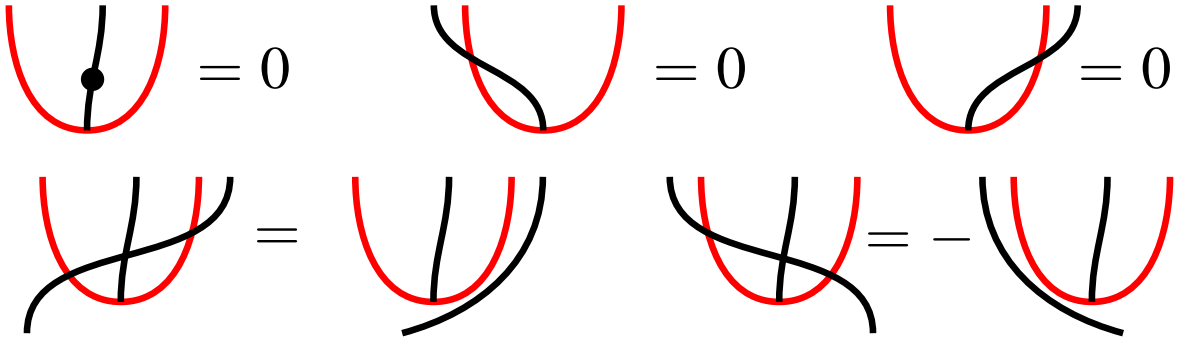


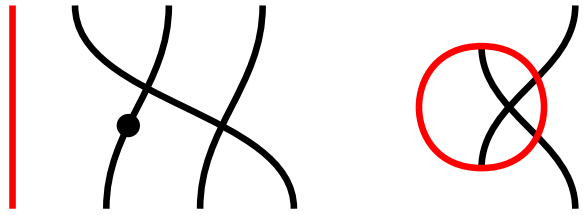
Of course, we need to have some relations:



With no other red lines, you get a $T^2 - T^0$ bimodule (i.e. a T^2-module). Of course, you get L.

The case of $\mathfrak{sl}_2$

So, for a circle, we get elements of the bimodule for each picture:



It's easy to check that we can simplify so that the bubble is separate.

But we have to think a bit harder than this; the functor for a cup isn't exact! You need to use a projective resolution!

The case of $\mathfrak{sl}_2$

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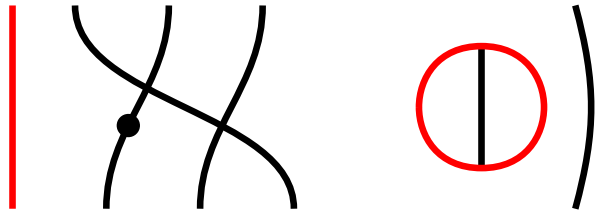

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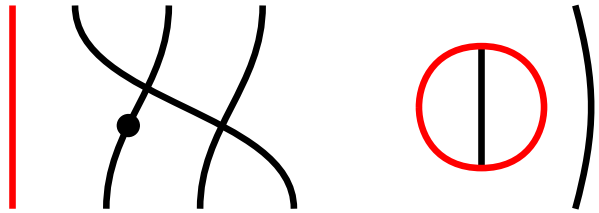

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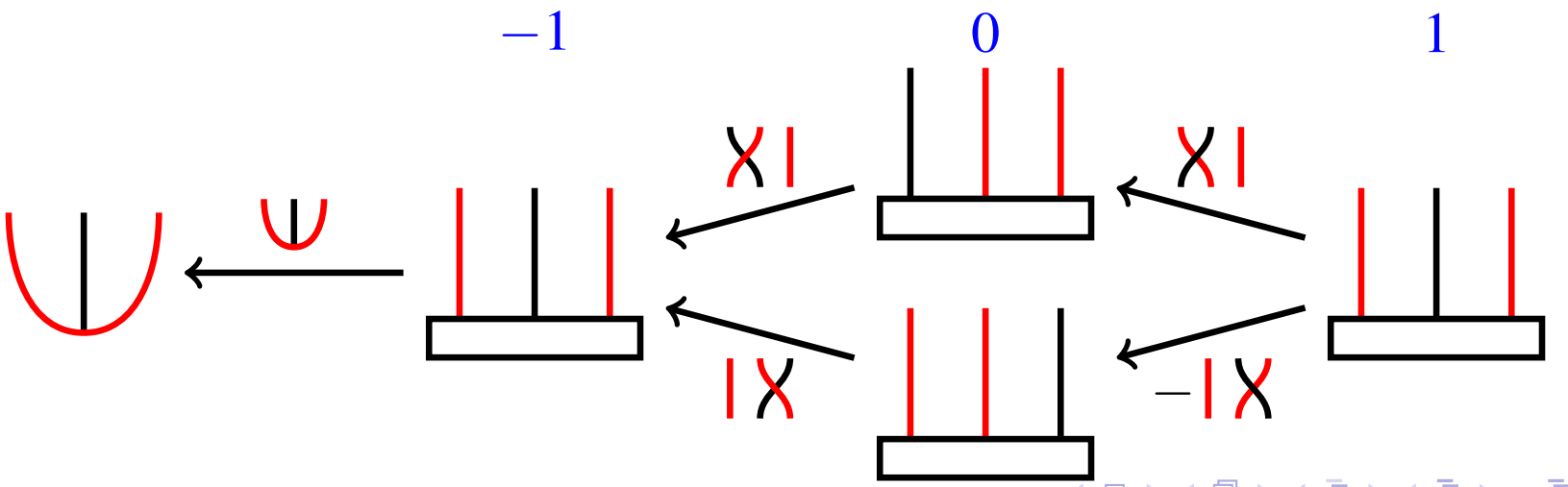
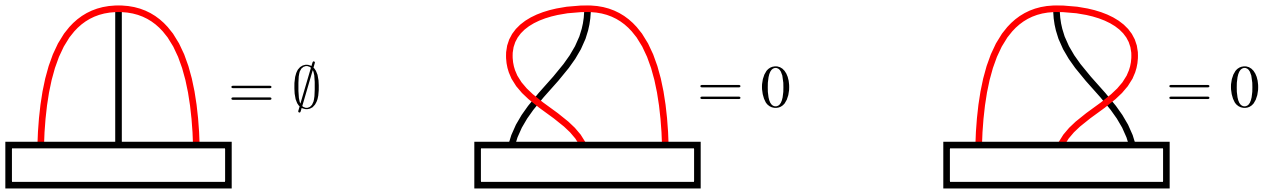


The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that



So, we can see the relations

$$-1 \qquad 0 \qquad 1$$

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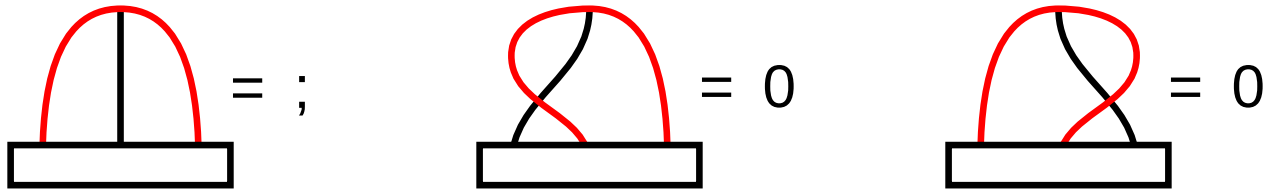
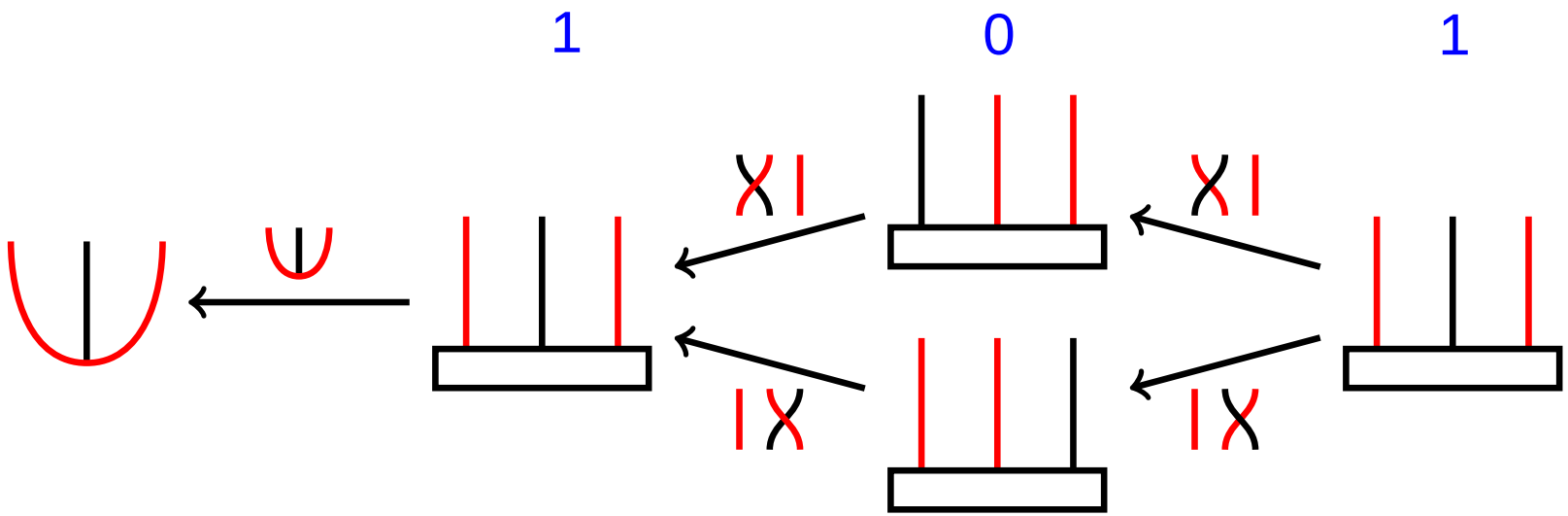
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