# Higher representation theory in algebra and geometry: Lecture V

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UVA

February 26, 2014

# References

For this lecture, useful references include:

- Webster, *Tensor product algebras, Grassmannians and Khovanov homology* (basically covers all this material, though in a slightly different presentation)
- Bar-Natan, *Khovanov's homology for tangles and cobordisms* (introduces $2-TL$ in a slightly more general form, and covers original definition of Khovanov homology)
- Khovanov, *A categorification of the Jones polynomial* and *A functor-valued invariant of tangles* (more detailed discussion of underlying topology)

The slides for the talk are on my webpage at:

`http://people.virginia.edu/~btw4e/lecture-5.pdf`

You can also find some proofs that I didn't feel like going through in class at:

`https://pages.shanti.virginia.edu/Higher_Rep_Theory/`

# Reminder from last time

Last time, we talked about how to categorify tensor products, inspired by the geometry of Schubert cells.

Tensor products play an important role in the theory of quantum groups, but for me, they're especially important because of connections to topology.

Today, I want to talk about these applications and how to categorify them.

# Why tensor?

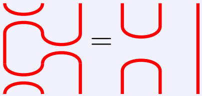
## Definition

*The* **Temperley-Lieb** *algebra $TL_n$ is generated elements $b_i$ for $i = 1, \dots, n-1$ subject to the relations*

$$b_i^2 = (q + q^{-1})b_i \qquad b_i b_{i\pm 1} b_i = b_i$$

$b_i =$ 

 $= q + q^{-1}$



# The Temperley-Lieb algebra

Consider $V_1 = \mathbb{C}^2[q, q^{-1}]$ spanned by $\{v_\pm\}$ with

$$Fv_+ = v_- \qquad Fv_- = 0 \qquad Ev_- = v_+ \qquad Ev_+ = 0.$$

Remember, quantization changes how we act on tensor products,

$$E(v \otimes w) = Ev \otimes w + q^{\mathrm{wt}(v)} v \otimes Ew \qquad F(v \otimes w) = q^{-\mathrm{wt}(w)} Fv \otimes w + v \otimes Fw.$$

$$F(v_+ \otimes v_+) = v_+ \otimes v_- + q^{-1} v_- \otimes v_+$$

$$F(v_+ \otimes v_-) = qv_- \otimes v_- \qquad F(v_- \otimes v_+) = v_- \otimes v_-$$

On $V_1 \otimes V_1$, we have an endomorphism $b$ is $q + q^{-1}$ times projection to invariants.

Since we're working over $U_q(\mathfrak{sl}_2)$, the meaning of invariants has changed.

$$b(v_+ \otimes v_-) = q^{-1} v_+ \otimes v_- - v_- \otimes v_+ \qquad b(v_- \otimes v_+) = qv_- \otimes v_+ - v_+ \otimes v_-$$

# The Temperley-Lieb algebra

**Proposition**

*The algebra $TL_d$ acts faithfully on the tensor product $V_1^{\otimes d}$ sending $b_i \mapsto 1 \otimes \cdots \otimes b \otimes \cdots \otimes 1$. This is the full endomorphism algebra of $V_1^{\otimes d}$ is $TL_d$.*

Note that if we set $q = 1$, then the action of $b_i$ becomes $1 - (i, i+1)$, so this is just the action of $\mathbb{C}[S_n]$ (which isn't faithful).

There's a deformation of $\mathbb{C}[S_n]$ which matches up with Temperley-Lieb algebra; this is the (Iwahori-)Hecke algebra.

# The Temperley-Lieb category

Actually, we can think of all of these algebras together as a category *TL*.

- The objects of the category are positive integers $\mathbb{Z}_{\geq 0}$.
- The morphisms $n \to m$ are $\mathbb{C}(q)$-linear combinations of crossingless planar diagrams in $\mathbb{R} \times [0, 1]$ that meet $y = 0$ in $n$ points and $y = 1$ in $m$ (up to isotopy).
- composition is stacking, with every circle becoming multiplication by $q + q^{-1}$.

We have a functor from *TL* to $U_q(\mathfrak{sl}_2)$ -mod sending $d \mapsto V_1^{\otimes d}$.

# Jones polynomials

**Definition**

*Recall that a* $(n, m)$ **tangle** *is a non-intersecting collection of smooth curves in* $\mathbb{R}^2 \times [0, 1]$ *which meets* $z = 0$ *at* $(0, 1, 0), \ldots, (0, n, 0)$ *and* $z = 1$ *at* $(0, 1, 1), \ldots, (0, m, 1)$ *(as usual, considered up to isotopy).*

*A* **ribbon tangle** *is also equipped with a framing, parallel to the y-axis at* $z = 0, 1$.

We can think of (ribbon) tangles as morphisms in a category $\mathsf{tang}$ whose objects are the integers $\mathbb{Z}_{\geq 0}$, and whose morphisms $n \to m$ are $(n, m)$ ribbon tangles up to isotopy.

# Jones polynomials

For a representation $W$ of $U(\mathfrak{sl}_2)$, we have an action of $S_d$ on $W^{\otimes d}$; for $U_q(\mathfrak{sl}_2)$, this is no longer the case.

The flip map is no longer a map of representations. You now have to choose. Do you want $v_+ \otimes v_- \mapsto v_- \otimes v_+$ or $v_- \otimes v_+ \mapsto v_+ \otimes v_-$? You must choose since you can't have both...

Well, you can't have both simultaneously, at least.

**Theorem (Jones)**

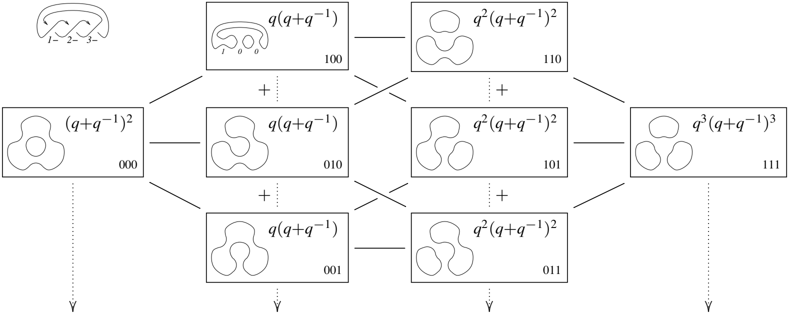
*There's a functor* tang $\to$ *TL which sends*


If we set $q = 1$, these are the same, and just reorder factors.

# Jones polynomials

In particular, every ribbon link has an associated polynomial, the **Jones polynomial**.

Usually people like to normalize this so it doesn't depend on the ribbon.




$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3 \tag{1}$$

$$= q^{-2} + 1 + q^2 - q^6 \tag{2}$$

# Categorified Jones polynomials

One obvious question is whether one can categorify this story. Luckily, the answer is yes!

So, what would that mean?

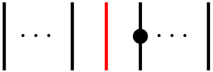
- Every vector space should turn into a category $V_1^{\otimes d} \mapsto T^d$-mod.
- Every morphism in the Temperley-Lieb category should turn into a functor, as should the maps we assigned to tangles.

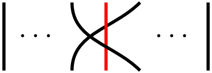
This is where we become glad that we invested in turning everything into algebras. If you really want we can discuss how to do this all using Whittaker sheaves, for the moment, I'm glad not to. Instead I just have to write down some bimodules.

# Diagrammatics

Let me just briefly remind you how these algebras work. They are built of diagrams with red and black strands.

The red strands correspond to the different tensor factors, and the black ones to lowering from the highest weight vector to lower ones.

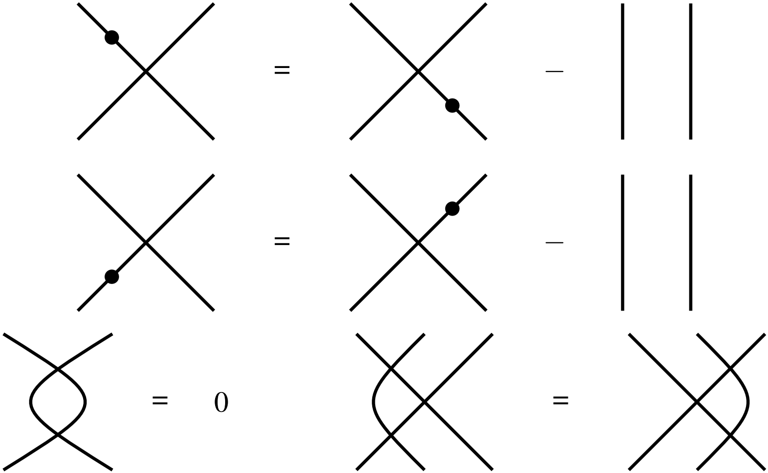
$y_i e_\kappa \mapsto$ 

$\psi_{i,\kappa,\kappa} \mapsto$ 

$\psi_{0,\kappa,\kappa'} \mapsto$ 

# Relations

The relations are easy to visualize.



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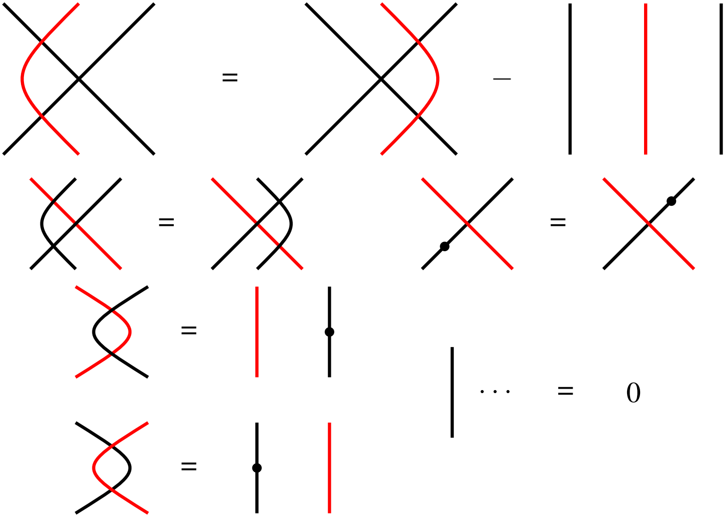

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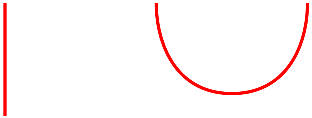

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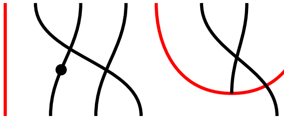

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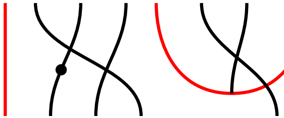
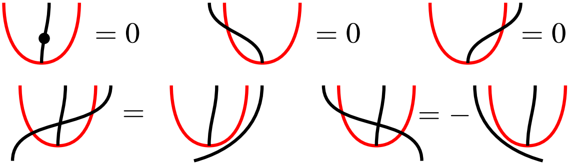


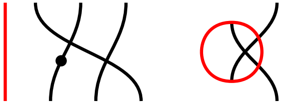
Of course, we need to have some relations:




With no other red lines, you get a $T^2 - T^0$ bimodule (i.e. a $T^2$-module). Of course, you get $L$.

## The case of $\mathfrak{sl}_2$

So, for a circle, we get elements of the bimodule for each picture:



It's easy to check that we can simplify so that the bubble is separate.

But we have to think a bit harder than this; the functor for a cup isn't exact! You need to use a projective resolution!

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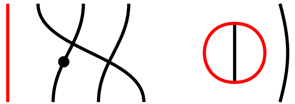

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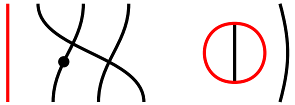

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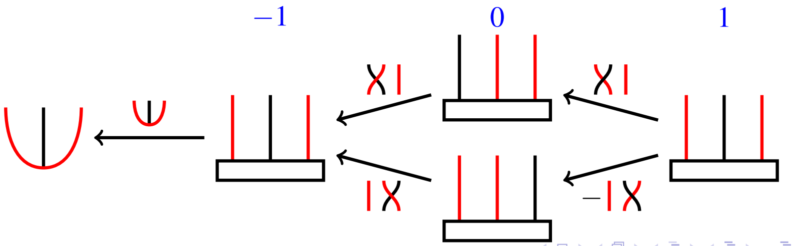

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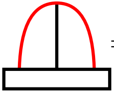
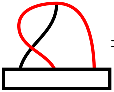
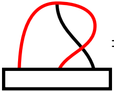
We can evaluate compositions by noting that

 $= \emptyset$  $= 0$  $= 0$

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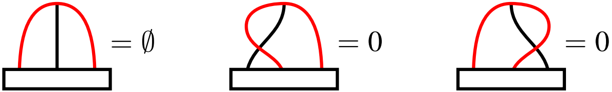


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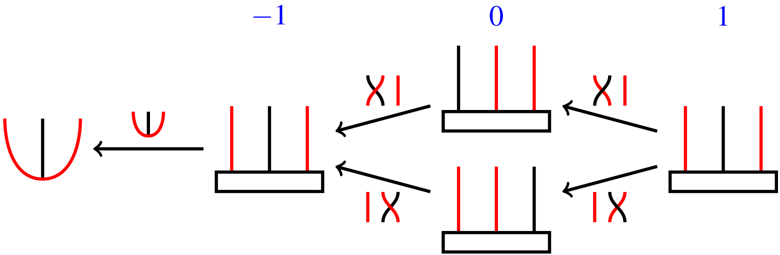

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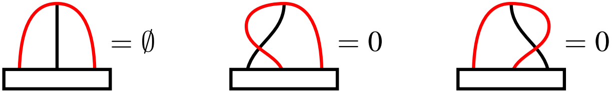


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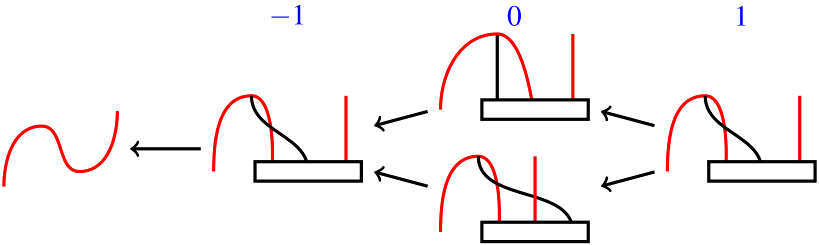

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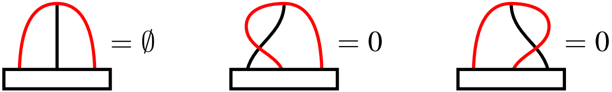


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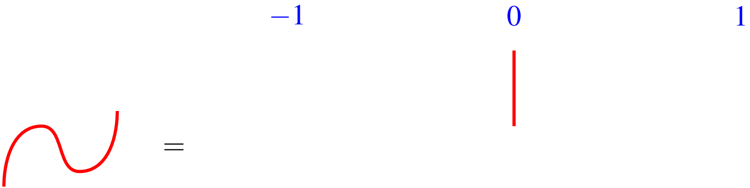

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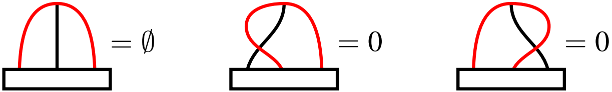


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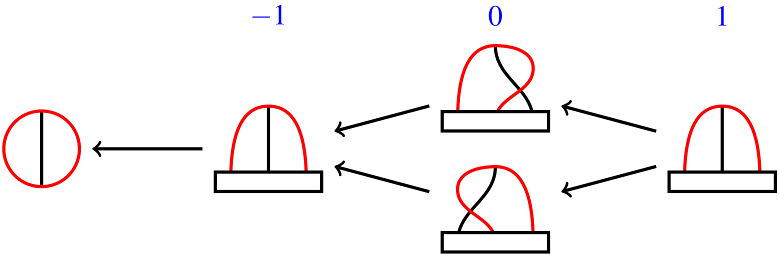

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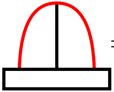
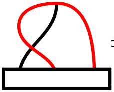
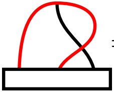
 $= \emptyset$  $= 0$  $= 0$

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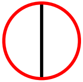
|  | $=$ | $\emptyset$ | $\oplus$ | $\emptyset$ |

## 2-Temperley-Lieb

I hope you all saw that last isomorphism, and thought “Ah-ha! The categorification of the Temperley-Lieb relations!”

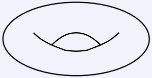
Then, I hope you said, “Wait, no, that’s not the right way to categorify relations. There has to be a reason behind the isomorphism.”

Indeed, we know how to do this. We have to upgrade the Temperley-Lieb picture by allowing morphisms between morphisms.

**Definition**

*The* **Temperley-Lieb** *2-category* $2-TL$ *has objects* $\mathbb{Z}_{\geq 0}$*, 1-morphisms formal sums of Temperley-Lieb diagrams and*

- *2-morphisms given by cobordisms between diagrams, modulo*

 $= 0$  $= 2$

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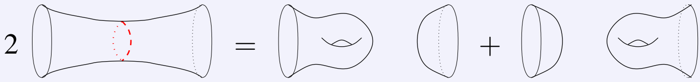

# The action of cobordisms

**Theorem (Chatav)**

*There is an representation of the Temperley-Lieb 2-category sending n to the derived category* $D^b(T^n\text{-gmod})$.

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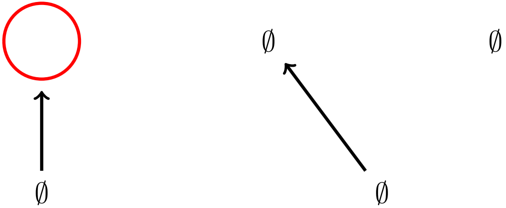


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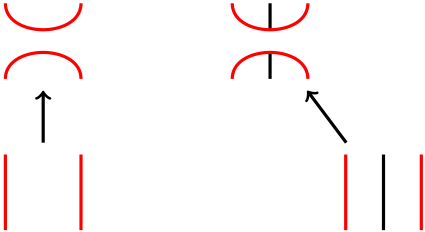

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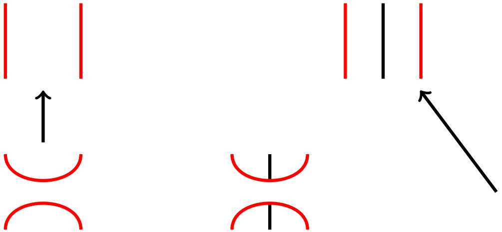

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# Tangles

Similarly, we can upgrade the category of tangles to a 2-category:

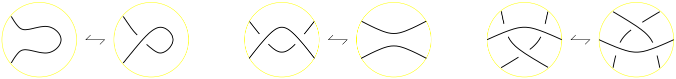
We let $2 - \mathsf{tang}$ denote the 2-category whose

- objects are $\mathbb{Z}_{\geq 0}$
- morphisms $m \to n$ are $(m, n)$-tangles
- 2-morphisms $T \to S$ are smooth cobordisms embedded in $(\mathbb{R}^2 \times [0, 1]) \times [0, 1]$ which match $T$ at $w = 0$ and $S$ at $w = 1$.

# Tangles

You can think of a smooth cobordism (after generic isotopy) as a sort of movie of tangle projections where at most times it's just isotoping without changing the picture, but at some times there's

- a Reidemeister move or



- a handle attachment (creating or destroying a circle, saddle cobordisms).

# The action of cobordisms

Having a representation of the category *TL* somewhere precisely means that we can construct the Jones polynomial in that world. This observation has a categorified analogue:

**Theorem (Bar-Natan; Clark-Morrison-Walker)**

*Consider the complexes*

*These define a 2-functor from* $2-\mathsf{tang}$ *to* $\mathcal{K}^b(2-TL)$.

To define this on cobordisms,

- for a handle attachement, we do the same on each term in complex
- for Reidemeister moves, we need to define by hand (and be very careful about signs!)

# Khovanov homology

It might seem like this is valued in some weird 2-category, but the endomophism category of 0 in $\mathcal{K}^b(2-TL)$ is just complexes of graded vector spaces up to homotopy.

**Theorem**

*This defines a bigraded vector space for each (ribbon) link called* **Khovanov homology**. *Computing Euler characteristic along one of the gradings gives the Jones polynomial.*

# Comparison to Khovanov homology

Applying Chatav's action, and passing to the derived category, we obtain that:

**Theorem**

*We have an action of* $2-\mathsf{tang}$ *on* $D^b(T^n\text{-gmod})$ *defined by complexes*

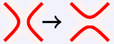

*This sends each link to the functor* $D^b(T^0\text{-mod}) \to D^b(T^0\text{-mod})$ *given by tensoring with the link's Khovanov homology.*

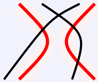
For me, the motivation is to have a way of thinking about Khovanov homology which explicitly uses the tensor product, since there's an analogue of the Jones polynomial for other Lie algebras which can only really be gotten at using the tensor product.

# Comparison to Khovanov homology

Let's try to be a bit more concrete about what this crossing is:

**Proposition**

*The induced map of bimodules*  *is surjective. Thus its cone is the kernel, the submodule of diagrams where we can see a pair of red lines with no black between them.*



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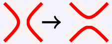
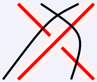
**Proposition**

*The induced map of bimodules*  *is surjective. Thus its cone is the kernel, the submodule of diagrams where we can see a pair of red lines with no black between them.*



*To remind that we have to go around, snap the red strands together.*

This second description will generalize to other Lie algebras. It also has a geometric description related to changing the standard flag you compute your Schubert cells with respect to.

# Khovanov homology

Khovanov homology is a perfect illustration of what categorifying buys you.

- It does distinguish some knots the Jones polynomial can't. In particular, the only knot with trivial Khovanov homology is the unknot.
- More importantly, because it's functorial, it allows us address new kinds of questions. For example, Khovanov homology supplies a new bound on the unknotting number of a knot.

It wasn't proven until the late 90's that you needed $(p-1)(q-1)/2$ crossing changes to untangle a $(p,q)$ torus knot, and the first proof involved horrible gauge theory. The proof using Khovanov homology is much easier.