Higher representation theory in algebra and geometry: Lecture VI

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References

For this lecture, useful references include:

- Chuang and Rouquier, *Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification* (introduces definition of categorical $\mathfrak{sl}_n$ actions via Hecke algebras)
- Khovanov and Lauda, *A categorification of quantum $\mathfrak{sl}(n)$* (covers connection to partial flag varieties in great detail)
- Brundan and Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras* (introduced grading on symmetric groups and affine Hecke algebras)
- B. W., *A note on isomorphisms between Hecke algebras* (contains proof we'll discuss today)

The slides for the talk are on my webpage at:

`http://people.virginia.edu/~btw4e/lecture-6.pdf`

You can also find some proofs that I didn't feel like going through in class at:

`https://pages.shanti.virginia.edu/Higher_Rep_Theory/`

Thus far...

We've talked a lot about categorical actions of $\mathfrak{sl}_2$. We've seen them show up in the representation theory of symmetric groups, in the geometry of Grassmannians and in the construction of knot invariants.

This leads us down a lot of interesting roads, but it's not very satisfying. After all, $\mathfrak{sl}_2$ is just the most basic of a big class of Lie algebras: the simple Lie algebras (or if you like to be fancier, the Kac-Moody algebras).

So, this is a natural direction to hunt. As with categorical $\mathfrak{sl}_2$-actions, the thing to look for is natural examples.

Simple Lie algebras

Recall, every simple Lie algebra over $\mathbb{C}$ (or more generally, Kac-Moody algebra) has a presentation by generators E_i, F_i, H_i for $i \in I$ with relations

$$[E_i, F_i] = H_i \qquad [H_i, E_i] = 2E_i \qquad [H_i, F_i] = -2F_i$$

$$\mathrm{ad}_{E_i}^{-a_{ij}+1} E_j = 0 \qquad \mathrm{ad}_{F_i}^{-a_{ij}+1} F_j = 0 \qquad [E_i, F_j] = 0 \qquad [H_i, H_j] = 0$$

for some Cartan matrix $A = (a_{ij})$.

The most popular source of these is from graphs, where I is the vertex set, and $-a_{ij}$ is the number of edges joining i and j.

The algebra $\mathfrak{sl}_{n+1}$ corresponds to a linear graph with n nodes.

Examples

Theorem (Beilinson-Lusztig-Macpherson)

There is a natural $U_{\sqrt{p}}(\mathfrak{sl}_n)$ action on the space of functions on the set of all $n-1$-step flags in a fixed vector space S, such that:

- *the maps E_i and F_i are induced by the pullback and pushforward are induced by the correspondence*

$$\mathsf{Gr}(\cdots \subset k_i \subset \cdots, S) \xleftarrow{\pi_2} \mathsf{Gr}(\cdots \subset k_i - 1 \subset k_i \subset \cdots, S) \xrightarrow{\pi_1} \mathsf{Gr}(\cdots \subset k_i - 1 \subset \cdots, S)$$

- *the functions concentrated on $\mathsf{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1, S)$ are of weight $(d-k_1, k_1-k_2, \dots, k_{n-1})$.*
- *the functions constant on each $\mathsf{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1, S)$ form a copy of the simple representation $\mathrm{Sym}^d(\mathbb{C}^n)$.*

Symmetric groups

Fix a field $\Bbbk$, and let $K = \oplus_m K(\Bbbk[S_m]\text{-proj})$ be the Grothendieck group of projective $\Bbbk[S_m]$ modules.

Earlier, we defined functors $\mathcal{F}_a$ and $\mathcal{E}_a$ for $a \in \mathbb{Z}$ by

$$\mathcal{E}_a(M) = \{h \in \mathrm{Res}^{S_m}_{S_{m-1}} M \mid (X_m - a)^N h = 0 \text{ for } N \gg 0\}.$$

$$\mathcal{F}_a(M) = \{s \otimes h \in \mathrm{Ind}^{S_m}_{S_{m-1}} M \mid s(X_m - a)^N \otimes h = 0 \text{ for } N \gg 0\}.$$

The functors $\mathcal{E}_i$ and $\mathcal{F}_i$ induce operators E_i and F_i on K.

Proposition

Each (E_i, F_i) generate a copy of $\mathfrak{sl}_2$, and

$$[E_i, [E_i, E_{i\pm 1}]] = 0 \qquad [E_i, E_j] = 0 \quad (i \neq j \pm 1)$$

$$[E_i, F_j] = 0 \quad (i \neq j) \qquad [F_i, [F_i, F_{i\pm 1}]] = 0 \qquad [F_i, F_j] = 0 \quad (i \neq j \pm 1)$$

Symmetric groups

That is, we have an action of the Lie algebra whose Dynkin diagram has vertices given by the image of $\mathbb{Z}$ in $\Bbbk$ where i and $i+1$ are adjacent.

- When $\mathrm{char}(\Bbbk) = 0$, then this is $\mathfrak{sl}_\infty$.
- When $\mathrm{char}(\Bbbk) = p$, then this is the affine Lie algebra $\widehat{\mathfrak{sl}}_p$.

Proposition

The Grothendieck group K is an irreducible representation of $\mathfrak{sl}_\infty$ or $\widehat{\mathfrak{sl}}_p$, generated by a highest weight vector satisfying $H_i v = \delta_{i,0} v$ ("the basic representation").

Categorifications of $\mathfrak{sl}_n$

These examples suggest there's some underlying structure here, so let's find it. The place that's most conducive to doing the calculation is on flag varieties. Recall that:

Proposition

The cohomology $\mathsf{Gr}(k_{n-1} \subset k_{n-2} \subset \cdots \subset k_1)$ is isomorphic to $C_d^{k_{n-1},\dots,k_1}$, generated by $e_i(1, k_{n-1}), e_i(k_{n-1}+1, k_{n-2}), \dots, e_i(k_1+1, d)$ in the coinvariant algebra C_d.

Whatever a categorical action of $\mathfrak{sl}_n$ is, we have an example of one on the categories $\oplus_{\mathbf{k}} C_d^{k_{n-1},\dots,k_1}$ -mod with functors $\mathcal{E}_i$ and $\mathcal{F}_i$ given by the tensor products

$$\mathcal{E}_i = C_d^{\dots,k_i-1,k_i,\dots} \otimes_{C_d^{\dots,k_i,\dots}} - : C_d^{\dots,k_i,\dots}\text{-mod} \to C_d^{\dots,k_i-1,\dots}$$

$$\mathcal{F}_i = C_d^{\dots,k_i,k_i+1,\dots} \otimes_{C_d^{\dots,k_i,\dots}} - : C_d^{\dots,k_i,\dots}\text{-mod} \to C_d^{\dots,k_i+1,\dots}$$

Categorifications of $\mathfrak{sl}_n$

First of all, the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ for each i generate a categorical action of $\mathfrak{sl}_2$. That is $\mathcal{E}_i$ and $\mathcal{F}_i$ satisfy $\mathfrak{sl}_2$ relations up to isomorphism, are biadjoint, and $\mathcal{F}_i^m$ has an action of the nilHecke algebra.

Well, that's a good start, but presumably it's not enough; we need to have some kind of interaction between these different $\mathfrak{sl}_2$'s.

Thinking geometrically, $\mathcal{F}_i\mathcal{F}_j$ corresponds to tensor product with H^* of the space of triples of flags

$$X_{i,j} = \{(V_\bullet, V'_\bullet, V''_\bullet) \mid V_k \supset V'_k, \dim(V_k/V'_k) = \delta_{k,i}, \\ V'_m \supset V''_m, \dim(V'_m/V''_m) = \delta_{m,j}\}$$

If $i \neq j \pm 1$, then switching the order of i and j gives the same space (Exercise: write down the isomorphism), and so $\mathcal{F}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{F}_i$.

Categorifications of $\mathfrak{sl}_n$

However, if $j = i + 1$, it's a different story. We may as well forget all the other spaces in the flag, and assume that $n = 3$, $i = 1$ and $j = 2$. Thus, we have that

$$X_{1,2} = \{V_1 \supset V_2, V_1'' \supset V_2''\}$$

$$X_{2,1} = \{V_1 \supset V_1'' \supset V_2 \supset V_2''\}$$

Thus, we have an obvious inclusion map $X_{2,1} \to X_{1,2}$ and thus pullback and pushforward maps in cohomology.

These maps aren't isomorphisms. Instead, their composition in either direction is multiplication by the Euler class of the line bundle $\mathrm{Hom}(V_2/V_2'', V_1/V_1'')$ since $X_{2,1}$ is the vanishing set of a section of this bundle (the induced canonical map).

KLR algebras

In the case of $\mathfrak{sl}_2$, the structure of a categorical action is controlled by the nilHecke action on powers of the functor $\mathcal{F}$. What will this structure look like in the $\mathfrak{sl}_n$ case?

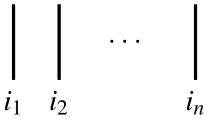
Consider the sum $\mathcal{F} := \oplus \mathcal{F}_i$. What acts on $\mathcal{F}^k$?

- there are idempotents $e_{\mathbf{i}}$ which project to $\mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1}$.
- the action of dots $y_1, \dots, y_k$ induced by the individual $\mathfrak{sl}_2$ actions.
- elements ψ_j which acts on $\mathcal{F}_{i_k} \cdots \mathcal{F}_{i_1}$ by
  1. the Demazure operator if $i_j = i_{j+1}$
  2. the pushforward by the map $X_{i_j+1,i_j} \to X_{i_j,i_j+1}$ if $i_{j+1} = i_j + 1$
  3. the pullback by the map $X_{i_j,i_j-1} \to X_{i_j-1,i_j}$ if $i_{j+1} = i_j - 1$
  4. the map induced the isomorphism $X_{i_{j+1},i_j} \cong X_{i_{j+1},i_j}$ if $i_{j+1} \neq i_j, i_j \pm 1$.

Diagrams

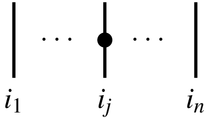
We represent these with diagrams much like before.

$\deg = 0$



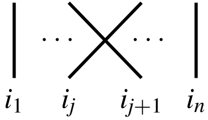
$e_{\mathbf{i}}$

$\deg = 2$



y_j

$\deg = -\langle \alpha_{i_j}, \alpha_{i_{j+1}} \rangle$

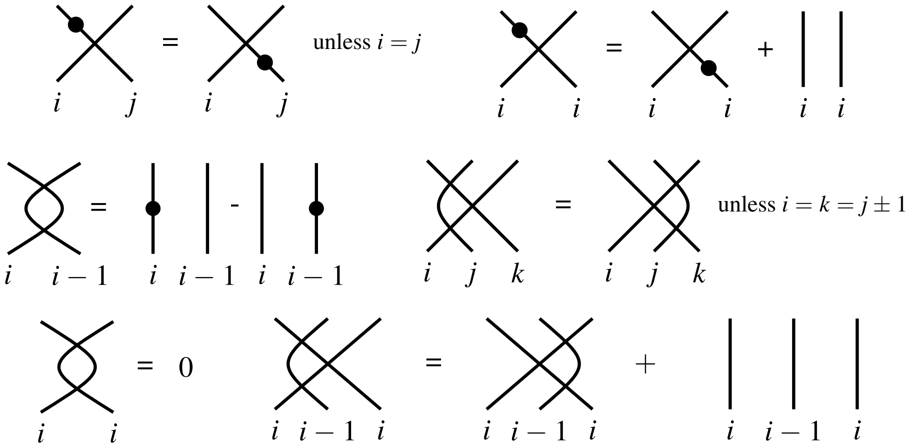


ψ_j

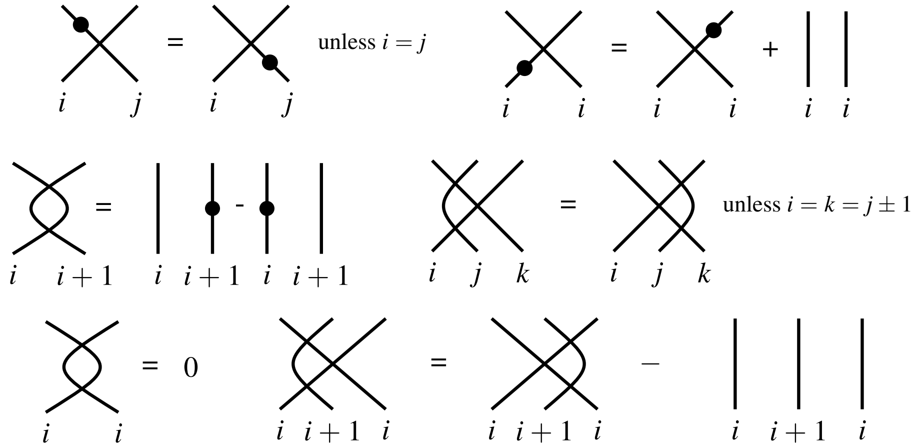
Definition

The **Khovanov-Lauda-Rouquier** *(KLR) or* **quiver Hecke** *algebra R_m for $\mathfrak{sl}_n$ is the algebra generated by these elements with m strands modulo the relations on the next slide.*

Diagrams



Diagrams



Categorical actions of $\mathfrak{sl}_n$

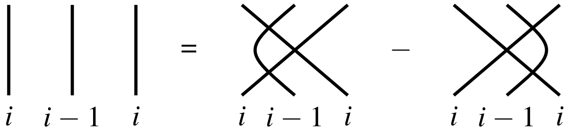
Definition

A categorical action of $\mathfrak{sl}_n$ is a collection of categorical $\mathfrak{sl}_2$ actions $(\mathcal{F}_i, \mathcal{E}_i)$ for $i = 1, \dots, n-1$, together with an extension of the nilHecke action to an action of R_m on $(\oplus \mathcal{F}_i)^m$.

We've deliberately set things up so we know at least one example of these: the modules over $C_d^{k_{n-1},\dots,k_1}$ for all different choices of $\mathbf{k}$.

Serre relations

Why do the Serre relations hold? Consider the relation



This is splitting of a sum as two orthogonal idempotents. In fact, this shows directly that:

$$\mathcal{F}_i\mathcal{F}_{i-1}\mathcal{F}_i \cong \mathcal{F}_i^{(2)}\mathcal{F}_{i-1} \oplus \mathcal{F}_{i-1}\mathcal{F}_i^{(2)}$$

which is just a rearranged Serre relation!

Proposition

Any categorical $\mathfrak{sl}_n$ action actually does induce an $\mathfrak{sl}_n$ action on the Grothendieck group. (Note, I haven't checked that $[E_i, F_j] = 0$.)

Categorical actions for other Lie algebras

It's relatively easy to guess how to extend this to the case of $\widehat{\mathfrak{sl}}_n$.

- You can construct a KLR algebra for $\widehat{\mathfrak{sl}}_n$ with the same sort of diagrams and relations, but with strands labeled by $\mathbb{Z}/n\mathbb{Z}$, rather that $\{1, \dots, n-1\}$.
- A categorical $\widehat{\mathfrak{sl}}_n$-action is a collection of $\mathfrak{sl}_2$ actions which have an action of this KLR algebra on $(\oplus \mathcal{E}_i)^m$.

Now, two things should bother you about this:

1. Many of you probably know that an $\widehat{\mathfrak{sl}}_n$ or $\mathfrak{sl}_n$ was defined in a different way in Chuang-Rouquier, using an affine Hecke algebra instead.
2. It's not clear that there are any examples of categorical $\widehat{\mathfrak{sl}}_n$-actions "in the wild" though you may recall that symmetric groups looked promising.

Hecke vs. KLR

There's an alternate definition of categorical actions via affine Hecke algebras.

Definition

The degenerate affine Hecke algebra h_m of rank m is generated by $t_1, \dots, t_{m-1}, x_1, \dots, x_m$ with relations:

$$t_i^2 = 1 \qquad t_i t_{i\pm 1} t_i = t_{i\pm 1} t_i t_{i\pm 1} \qquad t_i t_j = t_j t_i \ (i \neq j \pm 1)$$

$$x_i x_j = x_j x_i \qquad t_i x_i t_i = x_{i+1} - t_i \qquad x_i t_j = t_i x_j \ (i \neq j, j+1)$$

Definition

The affine Hecke algebra $H_m(v)$ of rank m is generated by $T_1, \dots, T_{m-1}$, $X_1^{\pm 1}, \dots, X_m^{\pm 1}$ with relations:

$$(T_i + 1)(T_i - v) = 0 \qquad T_i T_{i\pm 1} T_i = T_{i\pm 1} T_i T_{i\pm 1} \qquad T_i T_j = T_j T_i \ (i \neq j \pm 1)$$

$$X_i X_j = X_j X_i \qquad T_i X_i T_i = v X_{i+1} \qquad X_i T_j = T_i X_j \ (i \neq j, j+1)$$

Hecke vs. KLR

Consider a pair of adjoint functors $(\mathcal{F}, \mathcal{E})$ such that $\mathcal{F}^m$ carries an action of the affine Hecke algebra over $\Bbbk$ for some $v \in \Bbbk$. Let $U \subset \Bbbk$ be the union of the spectra of X acting on $\mathcal{F}$ (and that $\mathcal{F}(M)$ is the sum of its generalized eigenspaces). We give U a graph structure by adding an edge $u_1 \to u_2$ if $u_2 = vu_1$.

We can decompose $\mathcal{F} = \oplus_{u \in U} \mathcal{F}_u$ into its generalized eigenspaces for X, and similarly $\mathcal{E} = \oplus_{u \in U} \mathcal{E}_u$.

Alternate definition

If the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ define a weak $\mathfrak{sl}_2$ action, then the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ define an action of $\mathfrak{g}_U$, the algebra with Dynkin diagram U.

Since U is a union of linear and cyclic graphs, this just means a bunch of commuting $\widehat{\mathfrak{sl}}_r$ and $\mathfrak{sl}_s$ actions.

Hecke vs. KLR

Consider a pair of adjoint functors $(\mathcal{F}, \mathcal{E})$ such that $\mathcal{F}^m$ carries an action of the degenerate affine Hecke algebra over $\Bbbk$. Let $u \subset \Bbbk$ be the union of the spectra of x acting on $\mathcal{F}$ (and that $\mathcal{F}(M)$ is the sum of its generalized eigenspaces). We give u a graph structure by adding an edge $u_1 \to u_2$ if $u_2 = u_1 + 1$.

We can decompose $\mathcal{F} = \oplus_{u \in U} \mathcal{F}_u$ into its generalized eigenspaces for x, and similarly $\mathcal{E} = \oplus_{u \in U} \mathcal{E}_u$.

Alternate definition

If the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ define a weak $\mathfrak{sl}_2$ action, then the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ define an action of $\mathfrak{g}_U$, the algebra with Dynkin diagram U.

Since U is a union of linear and cyclic graphs, this just means a bunch of commuting $\widehat{\mathfrak{sl}}_r$ and $\mathfrak{sl}_s$ actions.

Hecke vs. KLR

For example, we have an action of h_m on $\mathrm{Ind}_{S_k}^{S_{k+m}}$ with

$$t_i \mapsto (i+k, i+k+1) \qquad x_i \mapsto \sum_{g<i} (g,i)$$

The spectrum u is the image of $\mathbb{Z}$ in $\Bbbk$; thus, as we saw before, it's a cycle if $\mathrm{char}(\Bbbk) = p$, and an infinite line if $\mathrm{char}(\Bbbk) = 0$.

Proposition

This defines a categorical $\widehat{\mathfrak{sl}}_p$ or $\mathfrak{sl}_\infty$-action on $\oplus_m \Bbbk[S_m]$-mod (according to our alternate definition).

Similarly, we'll get an $\widehat{\mathfrak{sl}}_p$-action if we look at representations of the (finite) Hecke algebra with $v \in \Bbbk$ a primitive pth root of unity. This is just one aspect of the analogy of this case with the case $\mathrm{char}(\Bbbk) = p$.

Hecke vs. KLR

So, why are these "the same thing"? Well, of course, it would be great if the (d)AHA were isomorphic to a KLR algebra, but a moment's thought shows this can't be so (for which graph?).

Proposition

Instead, there's a natural completion of $H_m(v)$ at the collection of quotients where X_i has spectrum U which is isomorphic to the KLR algebra R_m with graph U completed according to its grading.

For example, if we let U be the pth roots of unity in $\Bbbk$ of characteristic prime to p, we'll get the KLR algebra for $\widehat{\mathfrak{sl}}_p$.

Hecke vs. KLR

So, why are these "the same thing"? Well, of course, it would be great if the (d)AHA were isomorphic to a KLR algebra, but a moment's thought shows this can't be so (for which graph?).

Proposition

Instead, there's a natural completion of h_m at the collection of quotients where x_i has spectrum u which is isomorphic to the KLR algebra R_m with graph u completed according to its grading.

For example, if we let u be $\mathbb{Z}/p\mathbb{Z}$ in $\Bbbk$ of characteristic p, we'll get the KLR algebra for $\widehat{\mathfrak{sl}}_p$.

Polynomial representations

In order to think about KLR algebras, one of the most useful tools is a representation on a sum of polynomial rings. Let I be the vertex set of an oriented graph. Consider the sum of polynomial rings $\oplus_{\mathbf{i}\in I^m}\Bbbk[Y_1,\dots,Y_m]\epsilon_{\mathbf{i}}$. Consider the operators

$$\begin{aligned} e_{\mathbf{i}}\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} &= f(Y_1,\dots,Y_m)\delta_{\mathbf{i},\mathbf{j}}\epsilon_{\mathbf{j}} \\ y_k\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} &= Y_k f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} \\ \psi_k\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} &= \begin{cases} \frac{f-f^{s_i}}{Y_{k+1}-Y_k}\epsilon_{s_k\cdot\mathbf{j}} & j_k=j_{k+1} \\ (Y_{k+1}-Y_k)^{\#\{j_k\to j_{k+1}\}}f^{s_k}\epsilon_{s_k\cdot\mathbf{j}} & j_k\neq j_{k+1}\end{cases}\end{aligned}$$

Proposition

The algebra these generate is the KLR algebra of I.

Polynomial representations

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$$e_{\mathbf{i}}\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} = f(Y_1,\dots,Y_m)\delta_{\mathbf{i},\mathbf{j}}\epsilon_{\mathbf{j}}$$
$$y_k\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} = Y_k f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}}$$
$$\psi_k\cdot f(Y_1,\dots,Y_m)\epsilon_{\mathbf{j}} = \begin{cases} \frac{f-f^{s_i}}{Y_{k+1}-Y_k}\epsilon_{s_k\cdot\mathbf{j}} & j_k = j_{k+1} \\ P_{i_k i_{k+1}}(Y_{k+1},Y_k)f^{s_k}\epsilon_{s_k\cdot\mathbf{j}} & j_k \neq j_{k+1} \end{cases}$$

Proposition

The algebra these generate is the KLR algebra of I.

Generally, we can choose a Cartan matrix A and polynomials $P_{ij}(u,v)$ with

$$P_{ij}(u,v)P_{ji}(v,u) = t_{ij}u^{-a_{ji}} + \cdots + t_{ji}v^{-a_{ij}}.$$

Polynomial representations

The affine Hecke algebra also has a (signed) polynomial representation on $\Bbbk[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ given by

$$T_i F(X_1, \dots, X_m) = -F^{s_i} + (1-q)X_{i+1}\frac{F^{s_i} - F}{X_{i+1} - X_i}$$

We'll also complete this representation at the quotients where the spectrum of X_i has spectrum U. For each $\mathbf{u} = (u_1, \dots, u_m)$, we have the $\varepsilon_{\mathbf{u}}$ of 1 to the generalized u_i-eigenspace for X_i.

The idea of the isomorphism is to match up these two representations after completion. Fix any power series $B(z) = 1 + \sum_{i=1}^{\infty} B_i z^i$ with $B_1 \neq 0$.

$$X_1^{a_1} \cdots X_m^{a_m} \epsilon_{\mathbf{u}} \leftrightarrow (u_1 B(y_1))^{a_1} \cdots (u_m B(y_m))^{a_m} \varepsilon_{\mathbf{u}}$$

Polynomial representations

The degenerate affine Hecke algebra also has a (signed) polynomial representation on $\mathbb{k}[x_1, \dots, x_m]$ given by

$$t_i f(x_1, \dots, x_m) = -f^{s_i} - \frac{f^{s_i} - f}{x_{i+1} - x_i}$$

We'll also complete this representation at the quotients where the spectrum of x_i has spectrum u. For each $\mathbf{u} = (u_1, \dots, u_m)$, we have the $\varepsilon_{\mathbf{u}}$ of 1 to the generalized u_i-eigenspace for x_i.

The idea of the isomorphism is to match up these two representations after completion. Fix any power series $b(z) = b_1 z + \dots$ with $b_1 \neq 0$.

$$x_1^{a_1} \cdots x_m^{a_m} \epsilon_{\mathbf{u}} \leftrightarrow (b(y_1) + u_1)^{a_1} \cdots (b(y_m) + u_m)^{a_m} \varepsilon_{\mathbf{u}}$$

Polynomial representations

The trick is to consider the "intertwining elements"

$$\Phi_r := T_r + \sum_{\mathbf{u} \text{ s.t. } u_r \neq u_{r+1}} \frac{1 - qe^h}{1 - X_r X_{r+1}^{-1}} e_{\mathbf{u}} + \sum_{\mathbf{u} \text{ s.t. } u_r = u_{r+1}} e_{\mathbf{u}}$$

If you look at how this element acts in the polynomial representation, you can match it up with the element ψ_r:

$$\Phi_r \cdot F = \begin{cases} \frac{u_r B(y_r) - v u_{r+1} B(y_{r+1})}{u_{r+1} B(y_{r+1}) - u_r B(y_r)} F^{s_r} \epsilon_{\mathbf{u}^{s_r}} & u_r \neq u_{r+1} \\ \frac{B(y_r) - vB(y_{r+1})}{B(y_{r+1}) - B(y_r)} (F^{s_r} - F) \epsilon_{\mathbf{u}} & u_r = u_{r+1} \end{cases}$$

We then use the fact that $B(y_r) - aB(y_{r+1}) = (1-a) + B_1(y_r - y_{r+1}) + \cdots$ is invertible if $a \neq 1$, whereas $(B(y_r) - B(y_{r+1}))/(y_r - y_{r+1}) = B_1 + \cdots$ is well-defined and invertible.

Polynomial representations

The trick is to consider the "intertwining elements"

$$\phi_r := t_r + \sum_{\mathbf{u} \text{ s.t. } u_r \neq u_{r+1}} \frac{1}{x_i - x_{i+1}} e_{\mathbf{u}} + \sum_{\mathbf{u} \text{ s.t. } u_r = u_{r+1}} e_{\mathbf{u}}$$

If you look at how this element acts in the polynomial representation, you can match it up with the element ψ_r:

$$\Phi_r \cdot F = \begin{cases} \dfrac{b(y_r) + u_r - b(y_{r+1}) - u_{r+1} - 1}{b(y_{r+1}) + u_{r+1} - b(y_r) - u_r} F^{s_r} \epsilon_{\mathbf{u}^{s_r}} & u_r \neq u_{r+1} \\ \dfrac{b(y_r) - b(y_{r+1}) - 1}{b(y_{r+1}) - b(y_r)} (F^{s_r} - F)\epsilon_{\mathbf{u}} & u_r = u_{r+1} \end{cases}$$

We then use the fact that $b(y_r) - b(y_{r+1}) + a = a + b_1(y_r - y_r + 1) + \cdots$ is invertible if $a \neq 0$, whereas $(b(y_r) - b(y_{r+1}))/(y_r - y_{r+1}) = b_1 + \cdots$ is well-defined and invertible.

Polynomial representations

That is, we send

$$\Phi_r e_{\mathbf{u}} \mapsto \begin{cases} \dfrac{(B(y_r) - vB(y_{r+1}))(y_r - y_{r+1})}{B(y_{r+1}) - B(y_r)} \psi_r e_{\mathbf{u}} & u_r = u_{r+1} \\ \dfrac{B(y_r) - B(y_{r+1})}{(v^{-1}B(y_{r+1}) - B(y_r))(y_{r+1} - y_r)} \psi_r e_{\mathbf{u}} & u_r = vu_{r+1} \\ \dfrac{u_r B(y_r) - vu_{r+1} B(y_{r+1})}{u_{r+1} B(y_{r+1}) - u_r B(y_r)} \psi_r e_{\mathbf{u}} & u_r \neq u_{r+1}, vu_{r+1} \end{cases}$$

Theorem

This induces an isomorphism of completions between the KLR algebra R for U and $H_m(v)$.

Note that this isomorphism is *highly* non-unique. I kind of like to use $B(x) = e^x$, but Brundan and Kleshchev use $B(x) = 1 + x$.

Polynomial representations

That is, we send

$$\phi_r e_{\mathbf{u}} \mapsto \begin{cases} \dfrac{(b(y_r) - b(y_{r+1}) - 1)(y_r - y_{r+1})}{b(y_{r+1}) - b(y_r)}\psi_r e_{\mathbf{u}} & u_r = u_{r+1} \\ \dfrac{b(y_r) - b(y_{r+1})}{(b(y_{r+1}) - b(y_r) - 1)(y_{r+1} - y_r)}\psi_r e_{\mathbf{u}} & u_r = u_{r+1} + 1 \\ \dfrac{b(y_r) + u_r - b(y_{r+1}) - u_{r+1} - 1}{b(y_{r+1}) + u_{r+1} - b(y_r) - u_r}\psi_r e_{\mathbf{u}} & u_r \neq u_{r+1}, u_{r+1} + 1 \end{cases}$$

Theorem

This induces an isomorphism of completions between the KLR algebra R for u and h_m.

Note that this isomorphism is *highly* non-unique. I kind of like to use $B(x) = e^x$, but Brundan and Kleshchev use $B(x) = 1 + x$.

Gradings on $\Bbbk[S_m]$

One really interesting consequence of this isomorphism is that $\Bbbk[S_m]$ has an unexpected grading:

Proposition

The group algebra $\Bbbk[S_m]$ is the quotient of h_m by $x_1 = 0$. The other x_i's go to the Jucys-Murphy elements.

Thus, we can also describe $\Bbbk[S_m]$ as a quotient of the KLR algebra by the two sided ideal generated by $e_{\mathbf{i}}$ with $i_1 \neq 0$, and y_1.

For example, if $m = 2$, then as long as $p > 2$, this quotient is spanned by $e_{0,1}$ and $e_{0,p-1}$, the idempotents projecting to the trivial and sign reps. If $p = 2$, then these are the same, and $\Bbbk[S_2] = \Bbbk[y_2]/(y_2^2)$.

Gradings on $\Bbbk[S_m]$

If interpreted correctly, most natural modules over $\Bbbk[S_m]$ such as simples, Specht modules, permutation modules, and projectives can be graded.

Thus, interesting questions like:

(∗) what are the multiplicities of simples modules in Spechts/projectives?

have q-analogues where we consider graded multiplicities. These have, for example, interesting connections to Kazhdan-Lusztig polynomials.

To be precise, if one looks instead at the (finite) Hecke algebra over $\mathbb{C}$ at $v = e^{2\pi i/p}$, the graded multiplicities of simples in Specht modules will be affine parabolic Kazhdan-Lusztig polynomials. The multiplicities over $\mathbb{F}_p$ are not the same (though they are bounded below by the char 0 multiplicities) for all p, but for p in some range $c(m) \leq p \leq m$.

A long-standing conjecture of James had been that $c(m) \leq \sqrt{p}$. This was recently disproven by Williamson.