

# Higher representation theory in algebra and geometry: Lecture VII

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UVA

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# References

For this lecture, useful references include:

- Khovanov and Lauda, *A diagrammatic approach to categorification of quantum groups. I*
- Rouquier, *2-Kac-Moody algebras* (both present definitions of KLR algebra. I'll use style of former, and substance of latter)
- Lauda and Vazirani, *Crystals from categorified quantum groups* (introduced crystal structure on simples)
- Varagnolo and Vasserot, *Canonical bases and KLR-algebras* (exactly what the title says)
- Tingley and B.W., *Mirkovic-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras* (exactly what the title says)

The slides for the talk are on my webpage at:

<http://people.virginia.edu/~btw4e/lecture-7.pdf>

You can also find some proofs that I didn't feel like going through in class at:

[https://pages.shanti.virginia.edu/Higher\\_Rep\\_Theory/](https://pages.shanti.virginia.edu/Higher_Rep_Theory/)

# Cartan matrices

## Definition

A **symmetrizable Cartan matrix**  $A$  is a matrix with entries in  $\mathbb{Z}$  such that

- $a_{ii} = 2$ .
- $a_{ij} \leq 0$  and  $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ .
- there are relatively prime numbers  $d_i$  such that  $d_i a_{ij} = d_j a_{ji}$ .

This allows us to define a symmetric bilinear form on the formal span of symbols  $\{\alpha_i\}$  (the **root lattice**) via  $\langle \alpha_i, \alpha_j \rangle = d_i a_{ij} = d_j a_{ji}$ .

We'll also fix a field  $\mathbb{k}$  and polynomials  $Q_{ij}(u, v) \in \mathbb{k}[u, v]$  such that

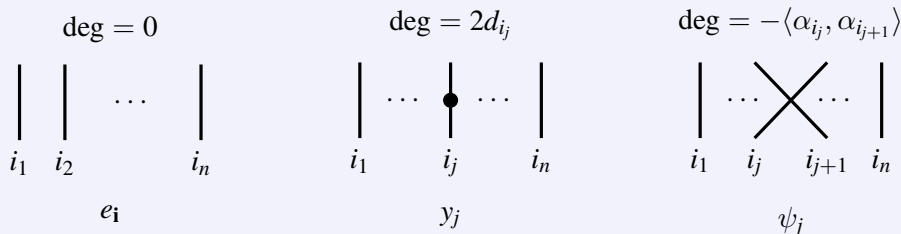
$$Q_{ij}(u, v) = t_{ij}u^{-a_{ij}} + \dots + t_{ji}v^{-a_{ji}}$$

We'll often want this to be homogeneous when  $\deg(u) = d_i$  and  $\deg(v) = d_j$ .

# KLR algebras

As discussed last time, attached to this data, we have a **Khovanov-Lauda-Rouquier** (KLR) or **quiver Hecke** algebra  $R_m$ .

The algebra  $R_m$  is generated over  $\mathbb{k}$  by these elements with  $m$  strands modulo the relations on the next slide.



# Diagrams

$$\begin{array}{c} \bullet \\ \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ j \end{array} = \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \bullet \\ \diagup \\ j \end{array} \quad \text{unless } i = j$$

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$$\begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ i \end{array} = 0 \quad \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ j \end{array} \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ i \end{array} = \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ j \end{array} \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ k \end{array} \begin{array}{c} \diagdown \\ i \end{array} - \boxed{\frac{Q_{ij}(y_3, y_2) - Q_{ij}(y_1, y_2)}{y_3 - y_1}} \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array} \begin{array}{c} | \\ i \end{array}$$

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# KLR algebras

This algebra is graded as long as  $Q_{ij}(u, v)$  is homogeneous via the rule

$$\deg \begin{array}{c} \times \\ i \quad j \end{array} = -\langle \alpha_i, \alpha_j \rangle \quad \deg \begin{array}{c} \bullet \\ | \\ i \end{array} = \langle \alpha_i, \alpha_i \rangle = 2d_i$$

Last time we defined a special case of this algebra in the context of categorical actions, and we can use this one similarly:

## Definition

*A categorical action of  $\mathfrak{g}$  is a collection of categorical  $\mathfrak{sl}_2$  actions  $(\mathcal{F}_i, \mathcal{E}_i)$  for  $i \in \Gamma$ , together with an extension of the nilHecke action to an action of  $R_m$  on  $(\oplus \mathcal{F}_i)^m$ .*

However, this algebra is worth studying on its own terms.

# Induction and restriction

## Definition

Let  $R_\nu$  be the subalgebra of diagrams where the sum of the labels on strands is  $\nu = \sum \nu_i \alpha_i$ .

We have an horizontal composition map  $R_\nu \otimes R_\mu \rightarrow R_{\nu+\mu}$ , and thus induction and restriction functors

$$M \circ N = R_{\mu+\nu} \otimes_{R_\nu \otimes R_\mu} M \times N \quad \text{Res}_{\nu'; \mu}^{\mu+\nu}(L) = \text{Res}_{R_\nu \otimes R_\mu}^{R_{\mu+\nu}}(L).$$

The functor  $\circ$  makes  $\bigoplus_\nu R_\nu\text{-gmod}$  into a monoidal category.

- The monoidal structure makes the Grothendieck group  $K = K^0(\bigoplus_\nu R_\nu\text{-gmod})$  into a ring:  $[M][N] = [M \circ N]$ , graded by  $\nu$ .
- Grading shift makes it into a  $\mathbb{Z}[q, q^{-1}]$ -module:  $[M(1)] = q[M]$ .
- The functor Res also endows  $K$  with a comultiplication:

$$\Delta(M) = \sum_{\nu'+\nu''=\nu} [\text{Res}_{\nu'; \nu''}^\nu(M)].$$



# Induction and restriction

## Proposition

$$[\mathrm{Res}_{\nu', \nu''}^{\nu}(M \circ N)] = \sum_{\mu = \mu' - \mathrm{wt}(M) + \mathrm{wt}(N)} q^{\langle \mu, \mu' \rangle} [\mathrm{Res}_{\mathrm{wt}(M) - \mu, \mu} M \circ \mathrm{Res}_{\mu', \mathrm{wt}(N) - \mu'} N]$$

The picture behind this is simple: you filter  $\mathrm{Res}_{\nu', \nu''}^{\nu}(M \circ N)$  by the number of strands that had pass from the left factor to the right (or vice versa).

This formula almost shows that  $\Delta$  is an algebra homomorphism; there's that annoying power of  $q$  in the way.

Actually,  $K$  is a Hopf algebra in the category of root lattice graded vector spaces with the braiding

$$\sigma(v \otimes w) = q^{\langle \mathrm{wt}(v), \mathrm{wt}(w) \rangle} w \otimes v.$$

# Induction and restriction

We know another such Hopf algebra,  $U_q^{\mathbb{Z}}(\mathfrak{n})$ , generated over  $\mathbb{Z}[q, q^{-1}]$  by the quantum divided powers  $F_i^{(n)} = \frac{F_i F_i^{(n-1)}}{q^{n-1} + q^{n-3} + \dots + q^{1-n}}$  modulo the quantum Serre relations

$$\sum_{i=0}^{-a_{ij}+1} (-1)^i F_i^{(i)} F_j F_i^{(-a_{ij}+1-i)}$$

with the coproduct  $\Delta(F_i) = F_i \otimes 1 + 1 \otimes F_i$ .

**Theorem (Khovanov-Lauda)**

$$K^0(\bigoplus_{\nu} R_{\nu}\text{-proj}) \cong U_q^{\mathbb{Z}}(\mathfrak{n}).$$

## Simples over KLRA

In particular, the classes of the simple modules give a basis of  $U_q(\mathfrak{n})$ , as do the classes of indecomposable projectives.

### Theorem (Varagnolo-Vasserot)

*If  $A$  is a symmetric Cartan matrix, the field  $\mathbb{k}$  has characteristic 0 and*

$$Q_{ij}(u, v) = (u - v)^{\#\{i \rightarrow j\}} (v - u)^{\#\{j \rightarrow i\}}$$

*then the classes of projectives (simples) give the (dual) canonical basis of  $U_q(\mathfrak{n})$ .*

Note, if any of these conditions fail, there are counter-examples. In fact, for  $A$  non-symmetric, the canonical basis doesn't have positive structure coefficients, which is obviously needed for the basis of projectives.

## Canonical bases

A few words about canonical bases. The algebra  $U_q(\mathfrak{n})$  has

- a unique semi-linear automorphism satisfying  $\bar{F}_i = F_i, \bar{q} = q^{-1}$ .
- a unique sesquilinear form satisfying

$$\langle vw, u \rangle = \langle v \otimes w, \Delta(u) \rangle \quad (F_i, F_j) = \delta_{ij} / (1 - q^{-2}).$$

These have categorical interpretations:

$$[\overline{M}] = [\mathrm{Hom}_R(M, R)] \quad \langle [M], [N] \rangle = \dim_q \mathrm{Hom}(M, N)$$

### Theorem (Lusztig)

Assume  $A$  symmetric. The canonical basis of  $U_q^{\mathbb{Z}}(\mathfrak{n})$  is the unique basis  $B$  with

- $\bar{b} = b$ ,
- multiplication has positive structure coefficients in this basis.
- $\langle b, b' \rangle \in \delta_{b,b'} + q^{-1}\mathbb{Z}[q^{-1}]$  (quasi-orthogonality),

# Canonical bases

- It's easy to work out that indecomposable projectives have a unique grading shift where  $P \cong \text{Hom}(P, R)$ .
- It's clear that the projectives will have positive structure coefficients since  $P \circ P'$  is a sum of projectives.
- Quasi-orthogonality is essentially equivalent to the Morita equivalence of  $R$  with a positively graded algebra  $A$  with  $A_0$  semi-simple.

This last item is by far the hardest (and the one that doesn't work if  $\text{char}(\mathbb{k}) \neq 0$  or  $A$  is non-symmetric).

At the moment, there's no purely algebraic argument I know, only a geometric one.

## Canonical bases

$$E_\nu = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\nu_j}) \quad G_\nu = \prod_i GL(\mathbb{C}^{\nu_i})$$

For a sequence  $\mathbf{i} \in \Gamma^n$ , let

$$Fl_{\mathbf{i}} = \{F_{i,1} \subset \cdots \subset F_{i,n} \subset \mathbb{C}^{\nu_i} \mid \dim(F_{i,k}/F_{i,k-1}) = \delta_{i,i_k}\}$$

$$X_{\mathbf{i}} = \{(x_\bullet, F_\bullet) \in E_\nu \times Fl_{\mathbf{i}} \mid x_e(F_{i,k}) \subset F_{j,k-1}\}$$

### Theorem (Varagnolo-Vasserot)

$e_{\mathbf{i}} R_n e_{\mathbf{j}} \cong H_*^{BM, G_\nu}(X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}})$ , *intertwining the product with convolution in homology.*

This is Morita equivalent to a positively graded algebra by the Decomposition theorem, which is sort of Hodge theory on steroids.

# Crystals

One of the interesting things about this theorem is that proving it doesn't really require indexing the basis, and indexing the canonical basis of  $U_q(\mathfrak{n})$  is tricky.

## Theorem (Kashiwara/Lusztig)

*The set of canonical basis vectors in  $U_q(\mathfrak{n})$  is endowed with a collection of **Kashiwara operators**  $f_i, f_i^*$  such that  $f_i b (f_i^* b)$  is the “leading term” of  $b F_i (F_i b)$ . The maps  $f_i$  and  $f_i^*$  are injective, and we let  $e_i, e_i^*$  be their (partially defined) inverse.*

We call such operators together with some other data (in particular, the weight function on basis vectors) a *bicrystal*. We let  $B(\infty)$  denote the crystal of canonical basis vectors in  $U_q(\mathfrak{n})$ .

# Crystals

## Theorem (Lauda-Vazirani)

*The simple modules over  $R_\nu$  for all  $\nu$  form a bicrystal isomorphic to  $B(-\infty)$ , where  $f_i L$  is the unique simple quotient of  $L \circ R_{\alpha_i}$  and  $f_i^* L$  the unique simple quotient of  $R_{\alpha_i} \circ L$ .*

This bijection is uniquely characterized by the fact that  $1 \mapsto R_0$ .

These crystal operators are simultaneously very simple, and devilishly tricky. It's easy to prove that there's a unique simple quotient, and very hard to actually construct the kernel or say what the dimension is.

Note that this statement doesn't depend at all on the field  $\mathbb{k}$  or the polynomials  $Q_{ij}$ . However, things like the dimension of the simples do!



# Crystals

It turns out that  $B(\infty)$  with Kashiwara operators shows up all over the place.

## Theorem (Kashiwara-Saito)

*If  $A$  is symmetric (i.e. comes from a graph), there is a canonical bijection between  $B(\infty)$  and the components of the moduli space of nilpotent representations of the corresponding preprojective algebra.*

## Theorem (Berenstein-Zelevinsky)

*If  $A$  is finite type, the set  $B(\infty)$  is in canonical bijection with the points of the group  $N$  of the semiring  $(\mathbb{Z}; \max, +)$ .*

## Theorem (Kamnitzer/Lusztig)

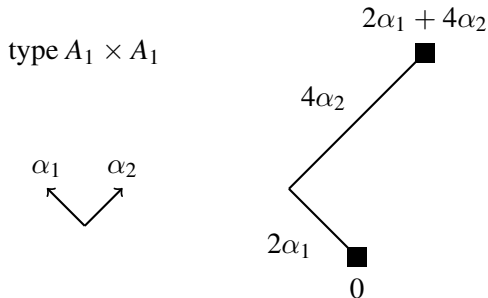
*If  $A$  is finite type, the set  $B(\infty)$  is in canonical bijection with a collection of polytopes called **Mirković-Vilonen** polytopes defined by combinatorial conditions on the structure of faces.*

# MV polytopes

MV polytopes are particularly interesting. We take their definition in rank 2 as an axiom:

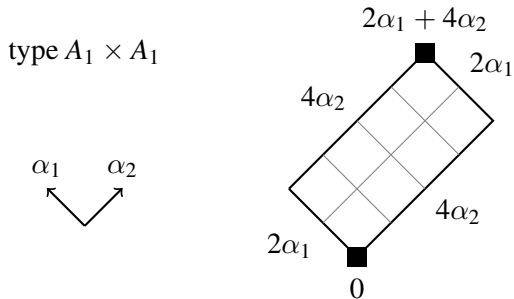
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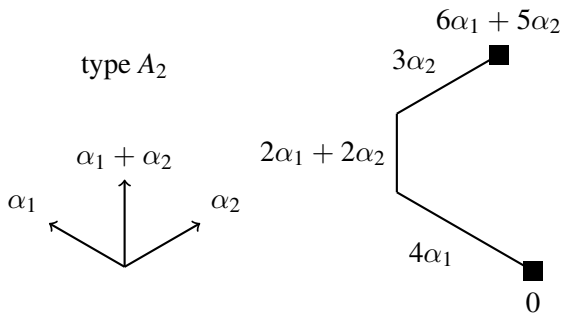
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a rectangle with sides parallel to simple roots

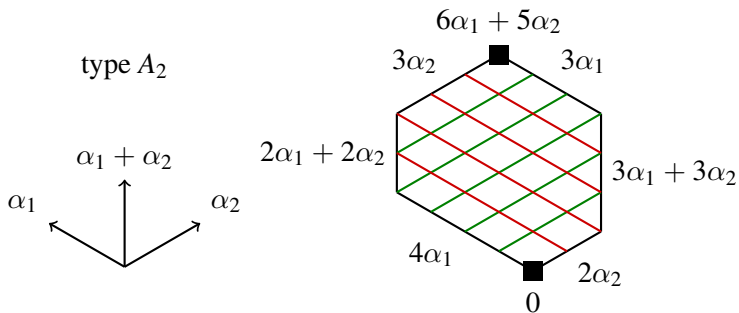
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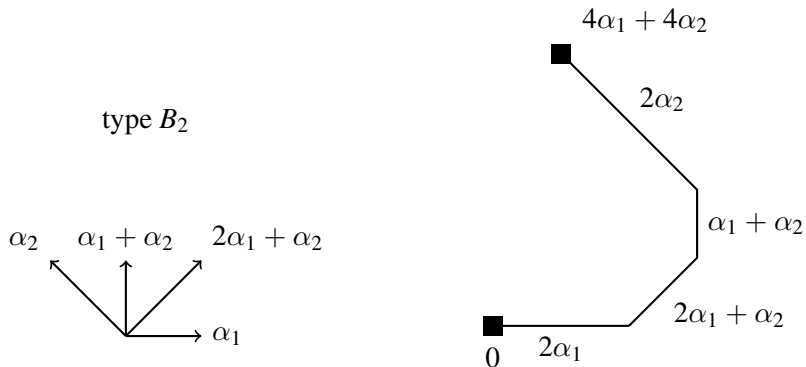
MV polytopes are particularly interesting. We take their definition in rank 2 as an axiom:



a hexagon with sides parallel to roots, at least one colored diagonal connects vertices.

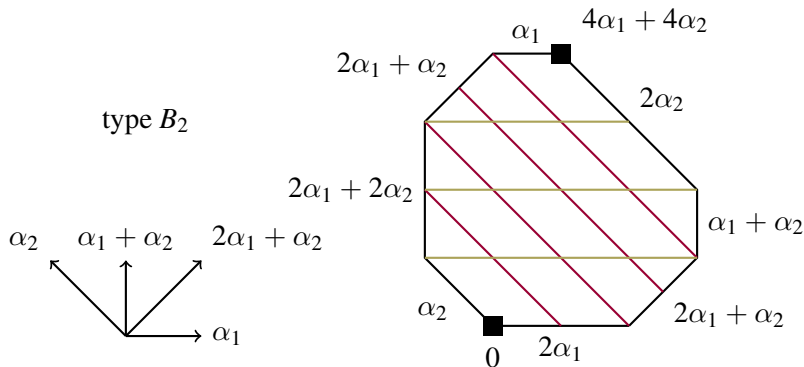
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MV polytopes are particularly interesting. We take their definition in rank 2 as an axiom:



a octagon with sides parallel to roots, two non-intersecting colored diagonals connect vertices (+ $\epsilon$ ).



# MV polytopes

For other semi-simple Lie algebras of rank at least 3, we can define MV polytopes inductively:

## Definition

A polytope whose faces are all parallel to faces of Weyl chambers in  $\mathfrak{h}^*$  is a **Mirković - Vilonen polytope** if each of its faces is an MV polytope for a lower rank Lie algebra.

Of course, it is enough to check this for 2-dimensional faces.

Depending on how you look at it, the bijection to  $B(\infty)$  is either dead easy, or completely unworkable. To apply  $f_i$ , you “just” hold all the stuff “below” the  $\alpha_i$ -edge from the top constant, extend that edge  $\alpha_i$ -further, and then fill in to get an MV polytope in the only way possible.

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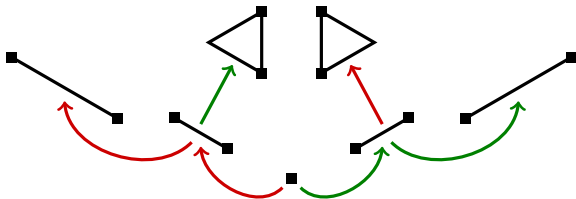
# MV polytopes for $\mathfrak{sl}_3$



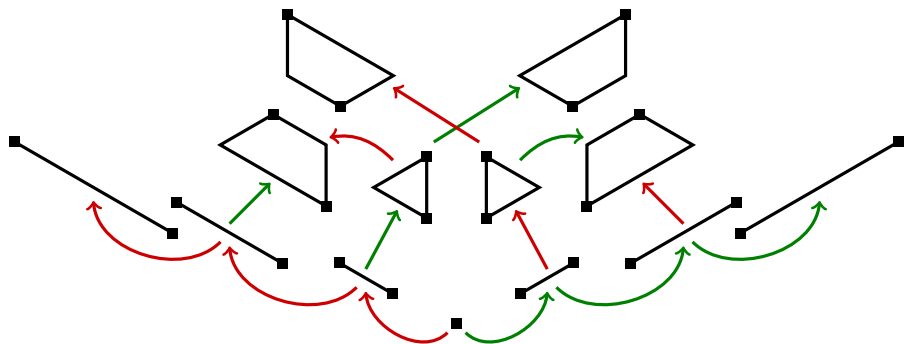
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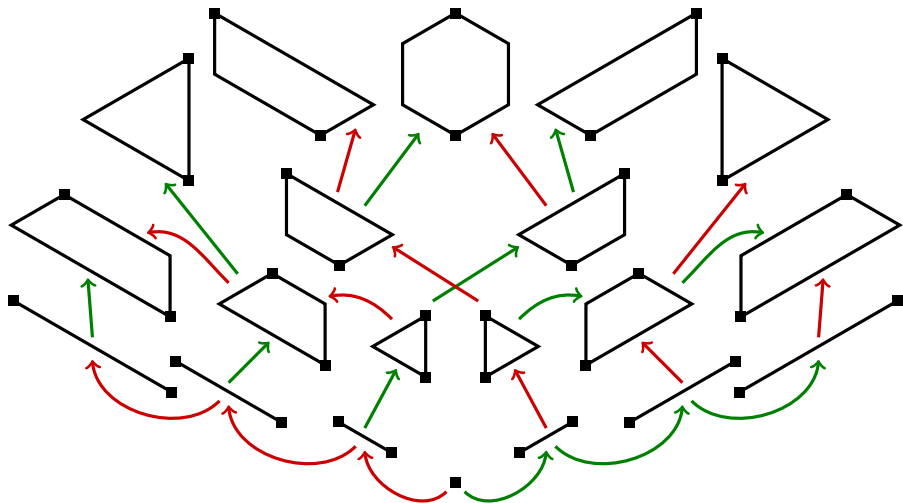
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# Crystals

So, MV polytopes and KLR simples give two different parameterizations of  $B(\infty)$ ; by abstract nonsense, there is a unique crystal isomorphism between them.



# Characters

One interesting question about KLR simples is their “character”

$$\chi_L = \sum_{\mathbf{i}} \dim_q e_{\mathbf{i}} L \cdot w[\mathbf{i}]$$

considered as a formal sum of words in the simple roots; this coincides with the “ $q$ -shuffle expansion” of  $[L]$ , thought of as an element of  $U_q(\mathfrak{n})$ .

Understanding these characters explicitly is a very interesting question; unfortunately, it’s a very hard one, much too hard for me.

## Definition

Let  $P_L$  be the convex hull of the points  $\nu'$  where  $\text{Res}_{\nu', \nu - \nu'} L \neq 0$ .

If you think of  $\mathbf{i}$  as a path, this is the convex hull of all the paths in the character.

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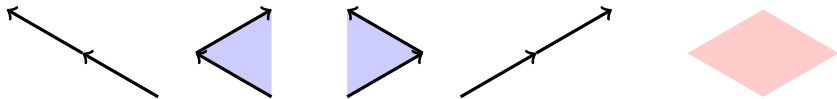
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# The polytope $P_L$

For example, for  $\mathfrak{sl}_3$ , the cases of  $R_0$  and  $R_{\alpha_i}$  are boring. For  $R_{2\alpha_i}, R_{\alpha_1+\alpha_2}$ , we get



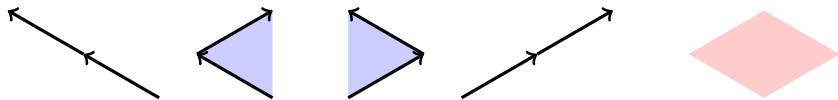
Now consider simples for  $R_{\alpha_1+2\alpha_2}$ .



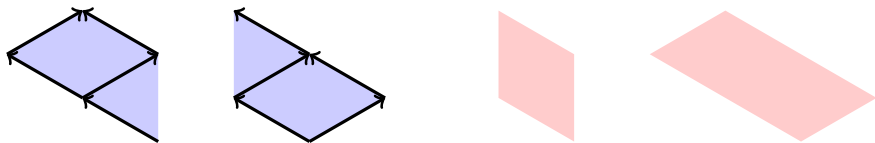
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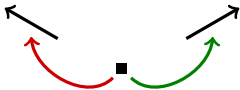


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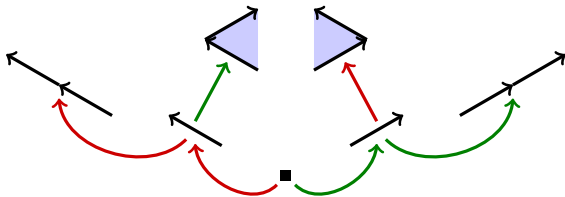
# KLR polytopes for $\mathfrak{sl}_3$



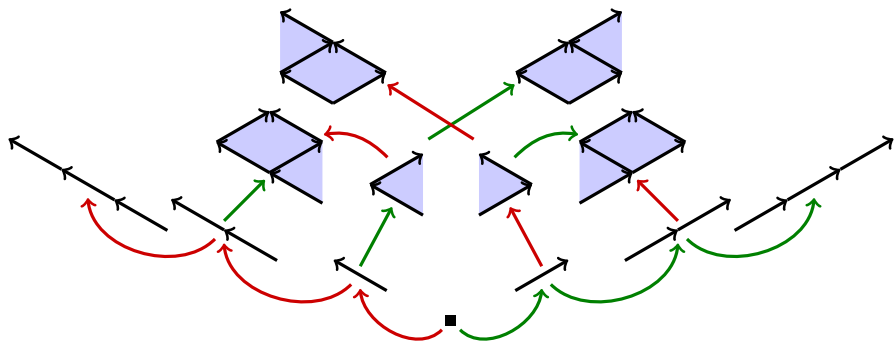
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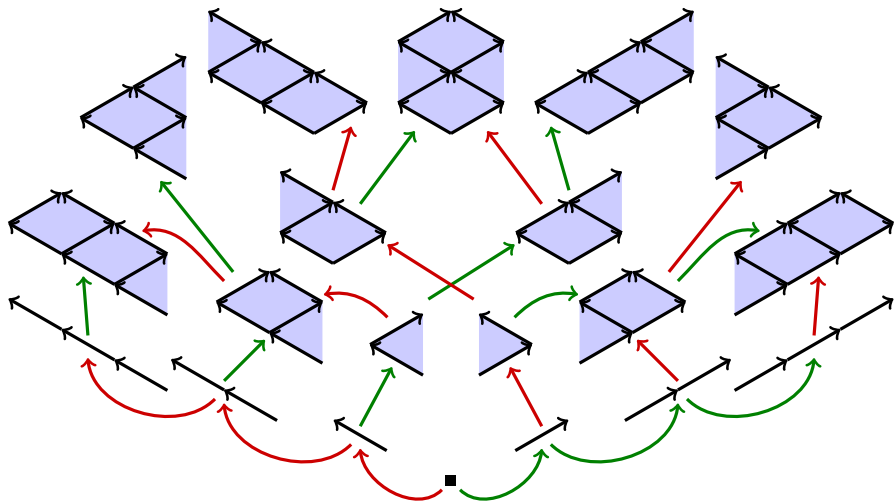


# KLR polytopes for $\mathfrak{sl}_3$





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## MV polytopes and $P_L$

### Theorem (Tingley-W.)

*The crystal isomorphism between KLRA simples and MV polytopes sends  $L \mapsto P_L$ .*

The proof is strikingly “high-level;” almost all you need is the crystal structure.

The extra leverage that the algebra gives you is that you can “multiply” simples using the induction functor (though sometimes you will fail and not get a unique answer) and the compatibility of this multiplication with crystal operations.

It seems likely that this is a purely combinatorial structure that can be formalized, which will make it easy to see how MV polytopes fall out the crystal  $(+\epsilon)$  structure alone.

## MV polytopes for other types

Now, cool as this theorem is, the really exciting part is that KLRA makes sense for all symmetrizable Kac-Moody algebras; the conventional MV polytopes do not. Thus, we can use  $P_L$  as the **definition** of MV polytopes for other types.

### Definition

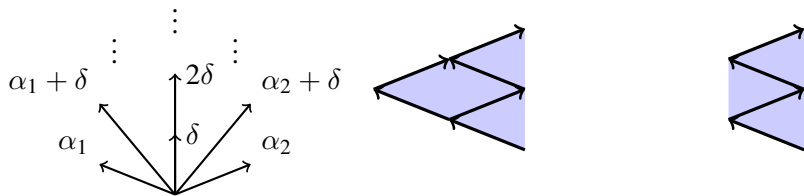
Let  $P_L$  be the convex hull of the points  $\nu'$  where  $\text{Res}_{\nu'; \nu - \nu'} L \neq 0$ .

Obviously, we're interested in how much of the finite type theory carries over:

- Do these polytopes uniquely determine the crystal element?
- Do they have a combinatorial characterization? That is, can we see on purely combinatorial grounds what these representations “look like?”

## MV polytopes for $\widehat{\mathfrak{sl}}_2$

Unfortunately, things don't seem to work quite as well any more; for  $\widehat{\mathfrak{sl}}_2$ , it seems our rule that MV polytopes are uniquely characterized by knowing one side of them seems to fail.



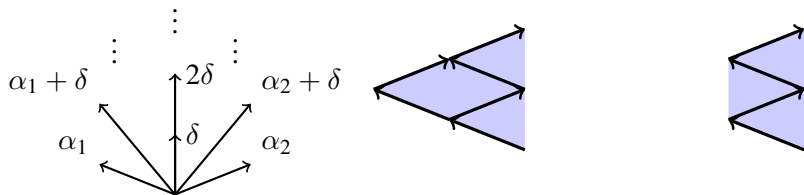
Even worse, there are different simples with the same polytope!

But these edges are not “the same;” one of them is forced to make contact with a word in the middle and the other isn't.

Baumann, Dunlap, Kamnitzer and Tingley suggest a solution: edges parallel to  $\delta$  must carry the extra data of a partition.

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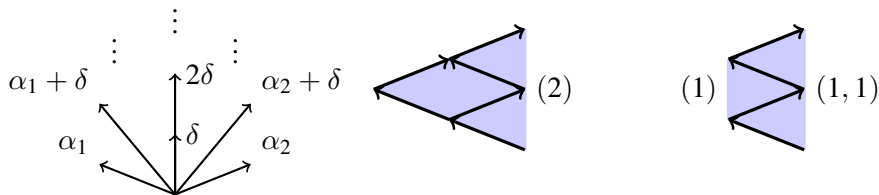
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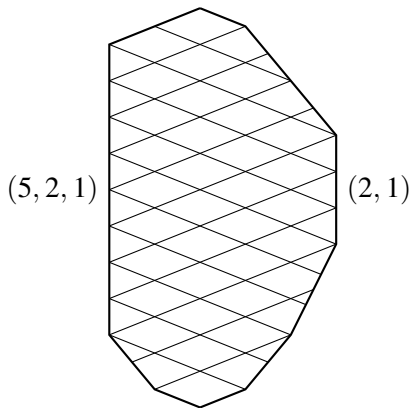
In our context, we look for an idempotent (i.e. a path) which touches that edge at top and bottom, but a minimal number of times in between; the attached partition is just the sizes of the gaps between touches.

### Theorem (Tingley-W.)

*The assignment  $L \rightarrow P_L$  enriched with this partition data is precisely the crystal isomorphism to  $\hat{\mathfrak{sl}}_2$  MV polytopes defined by Baumann, Dunlap, Kamnitzer and Tingley.*

## MV polytopes for $\hat{\mathfrak{sl}}_2$

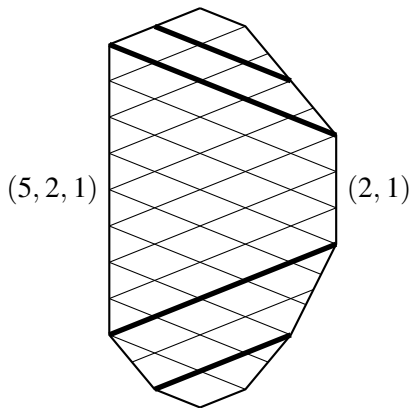
These are defined, as in finite type, in terms of non-intersecting diagonals along root directions.





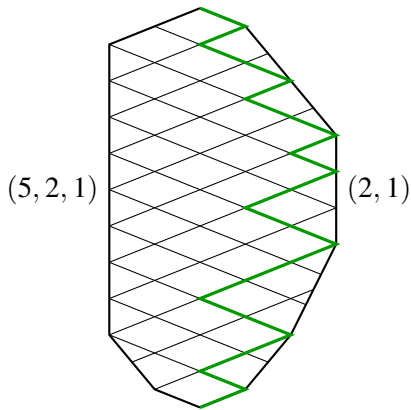
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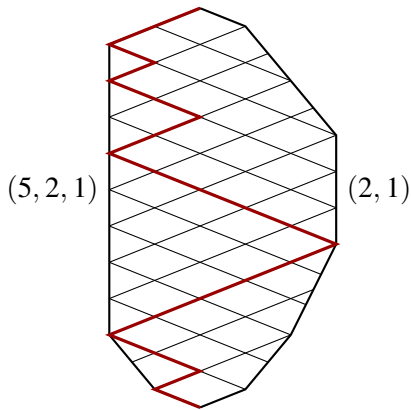
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## MV polytopes for affine $\mathfrak{g}$

For a general affine Lie algebra, one has to attach a partition to each facet parallel to  $\delta$ .

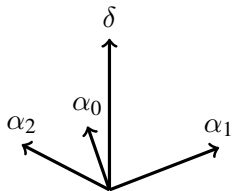
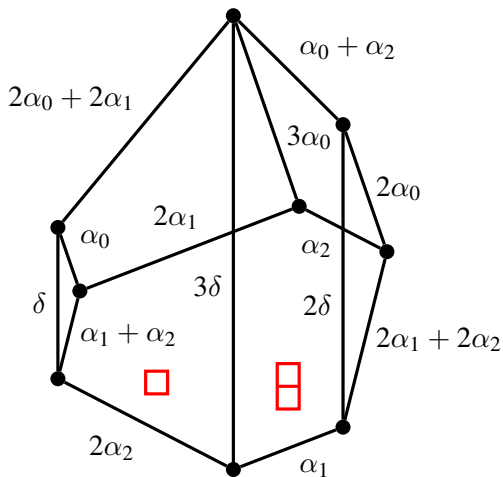
You should think of these as telling you the longest “leaps” you can take in the  $\delta$ -direction when you try to “walk” from the top of the facet to the bottom.

### Theorem?

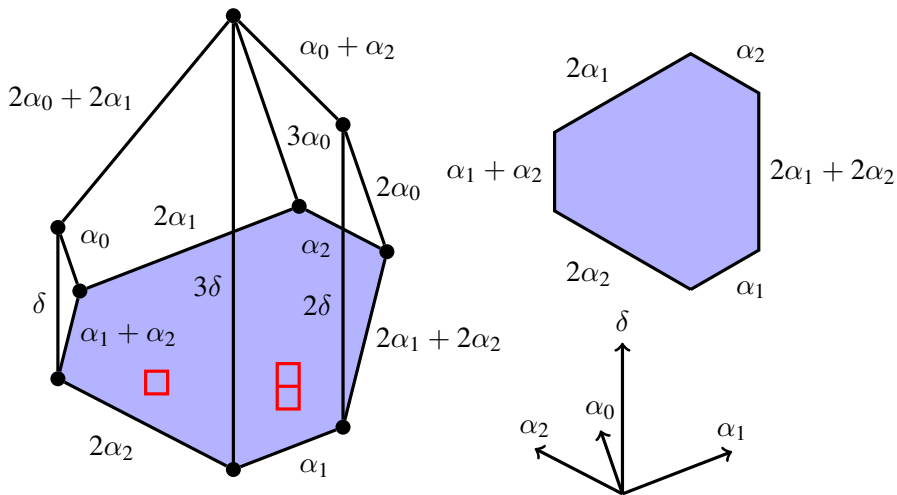
*The 2-dimensional faces of  $P_L$  are polytopes for KLR representations of a rank 2 algebra?*

The two imaginary edges of an affine 2-face come from the 2 facets that intersect it without containing it. So the edges should be labelled with those partitions. But they're too long!

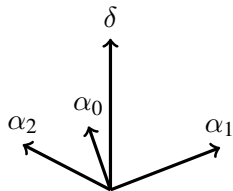
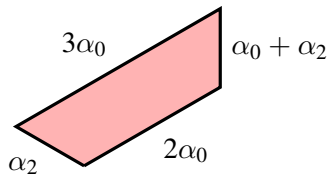
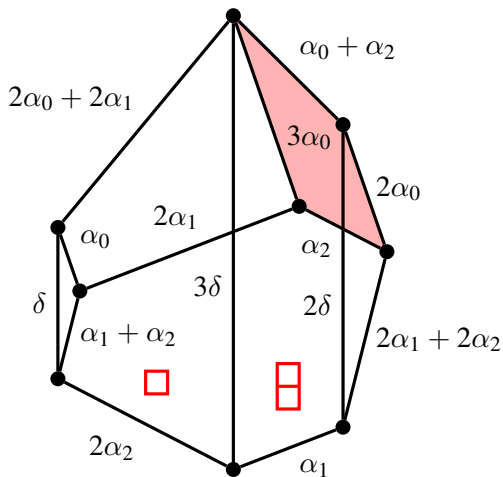
# An $\hat{\mathfrak{sl}}_3$ example



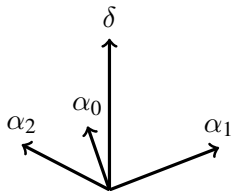
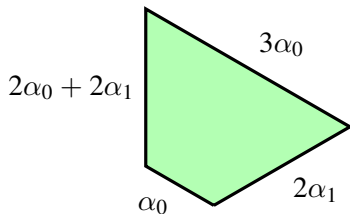
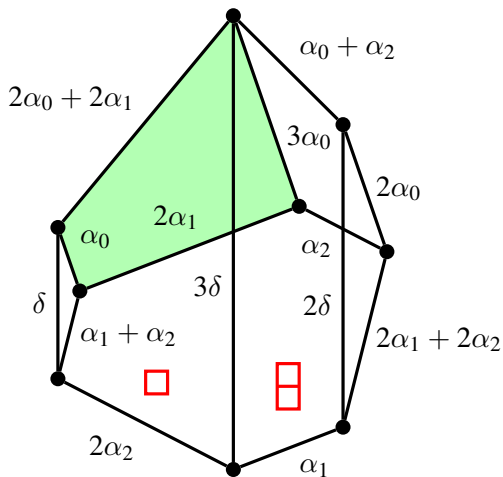
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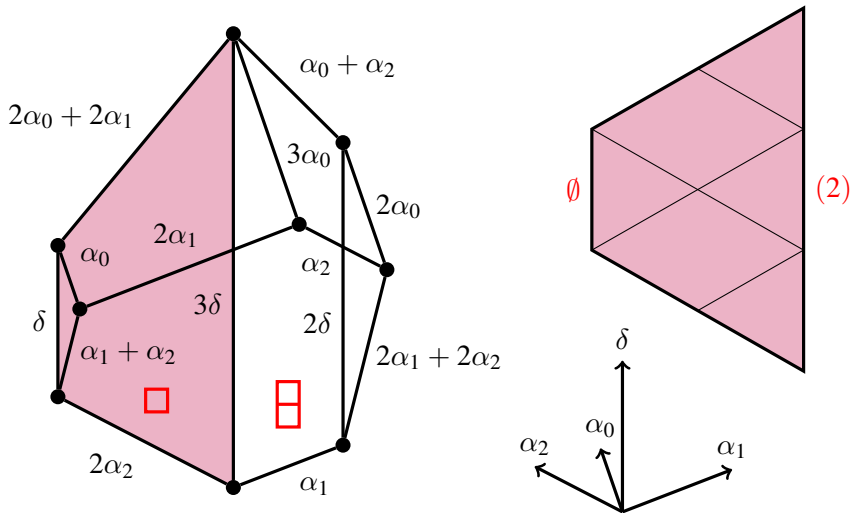


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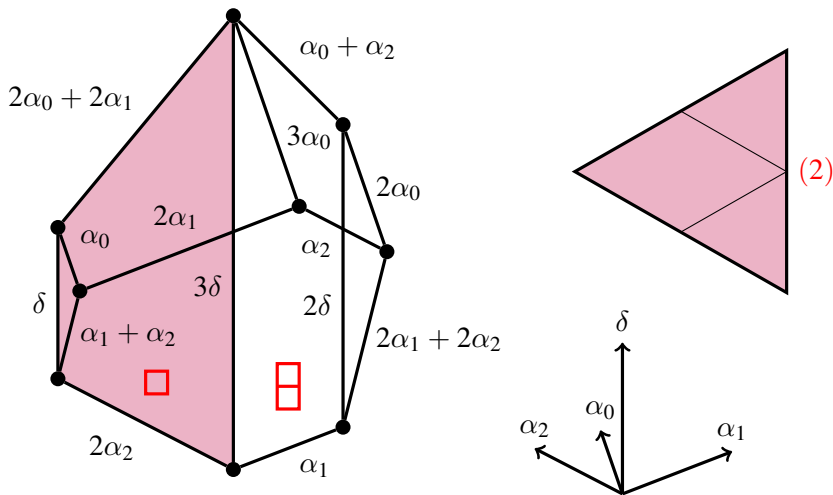




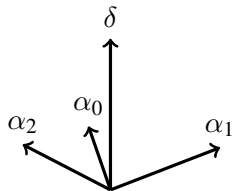
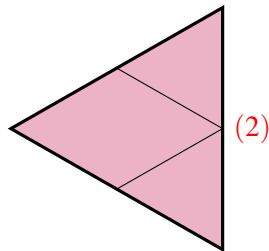
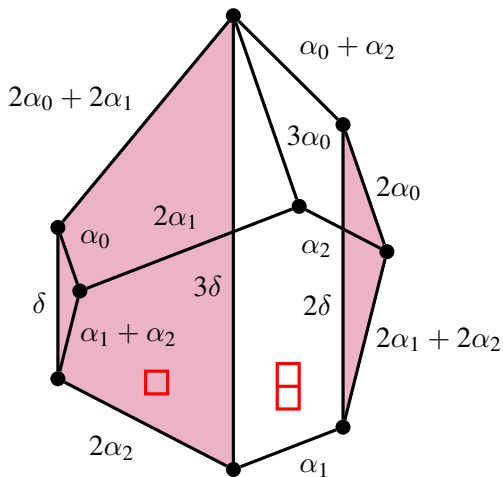
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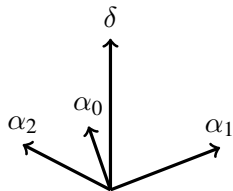
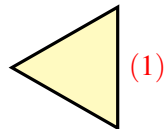
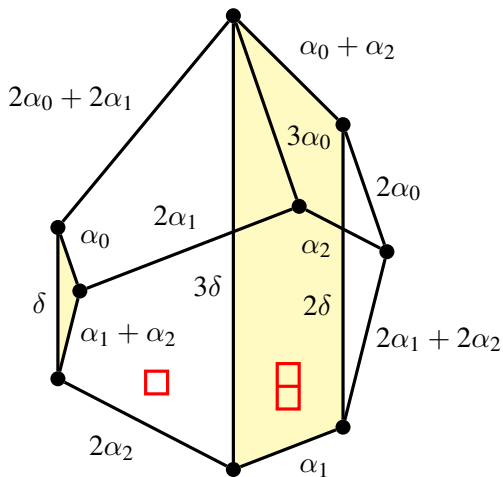
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# MV polytopes for affine $\mathfrak{g}$

## Theorem (Tingley-W.)

The polytopes  $P_L$  for  $\mathfrak{g}$  of affine type are uniquely determined by the property that any 2-face is

- a finite type rank 2 MV polytope if it's **real** (not parallel to  $\delta$ ) or
- the Minkowski sum of  $n\delta$  with an affine rank 2 MV polytope if it's **imaginary** (parallel to  $\delta$ ; here  $n$  is the number of boxes in the partitions labeling the facets containing that face).

## Corollary

If  $\mathfrak{g}$  has symmetric Cartan matrix, it coincides (after transposing partitions) with the affine MV polytopes defined using quiver varieties by Baumann, Kamnitzer and Tingley.

The transpose thing is silly; it's the question of whether you should index nilpotents by their Jordan type, or the type of the coarsest flag they preserve.

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# Labeling

In general, the edges of our polytope are labeled with *representations*; fixing one edge  $e$ , a simple  $L$  of weight  $\nu$  can be restricted to the subalgebra

$$R_{\alpha(e)} \otimes R_e \otimes R_{\nu - \omega(e)}.$$
$$\underbrace{\left| \cdots \right|}_{\alpha(e)} \quad \underbrace{\left| \cdots \right|}_{\text{wt}(e)} \quad \underbrace{\left| \cdots \right|}_{\omega(e)}$$

This restriction will be of the form  $L' \boxtimes L_e \boxtimes L''$  for simples of the smaller algebras. We associate  $L_e$  to the edge  $e$ .

This representation will always be “semi-cuspidal;” that is, its character stays inside the polytope. If  $e$  is always parallel to a real root, there is only one of these; if parallel to an imaginary, there can be many more.

# Labeling

In the finite type case, the convex paths from  $0$  to  $\nu$  encode previously known combinatorial information: the **Lusztig data**.

Each of these corresponds to a convex order on roots, and the lengths of the various edges describe how to get the canonical basis vector in question from a PBW basis constructed from that order.

## Theorem (Tingley-W.)

*The representation  $L$  is uniquely determined by  $P_L$  and the representations  $L_e$ ; in fact, it can be reconstructed from any convex path along edges from  $0$  to  $\nu$  with its labels.*

*Thus there exist Lusztig data for crystals in all types, with semi-cuspidal representations replacing the lengths of edges.*