Higher representation theory in algebra and geometry: Lecture VIII

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UVA

April 8, 2014

References

For this lecture, useful references include:

- B.W., *Knot invariants and higher representation theory*

The slides for the talk are on my webpage at:

`http://people.virginia.edu/~btw4e/lecture-8.pdf`

You can also find some proofs that I didn't feel like going through in class at:

`https://pages.shanti.virginia.edu/Higher_Rep_Theory/`

The future

So, there are 3 class meetings left. In what time is left, I want to try to cover two interesting applications of the theory we've discussed.

- the construction of knot invariants using this theory. We've already discussed one special case of this, using $\mathfrak{sl}_2$-categorifications to obtain the Jones polynomial. This generalizes to other types. There's also a "dual" construction of these knot invariants for $\mathfrak{sl}_n$, which we'll likely get to in Lecture 9. This also includes some interesting connections to algebraic geometry.
- the perspective on the representation theory of Cherednik algebras afforded by higher representation theory. This is is, of course, an enormous topic, but I think it's an exciting application of the theory, and one worth discussing a bit. I anticipate that will be Lecture 10.

Roadmap

quantum groups $U_q(\mathfrak{g})$

$\downarrow$

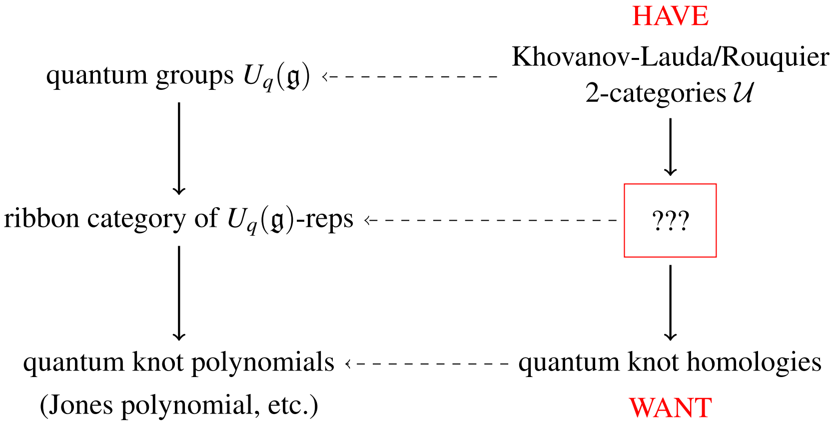
ribbon category of $U_q(\mathfrak{g})$-reps

$\downarrow$

quantum knot polynomials

(Jones polynomial, etc.)

Roadmap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Khovanov-Lauda/Rouquier
2-categories $\mathcal{U}$
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quantum knot polynomials
(Jones polynomial, etc.)
quantum knot homologies
WANT

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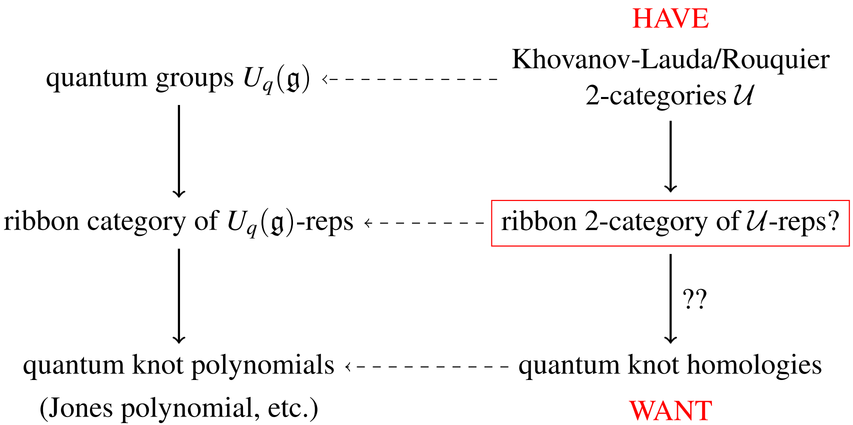
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↓		↓
ribbon category of $U_q(\mathfrak{g})$-reps	⇠ - - - -	categorifications of tensor products of simples
↓		↓ !
quantum knot polynomials (Jones polynomial, etc.)	⇠ - - - -	quantum knot homologies
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Reshetikhin-Turaev invariants

Let me briefly indicate how the left side of the diagram works.

Quantum groups are deformations of universal enveloping algebras. Perhaps the most important thing about them is that they deform the **tensor product** of $U(\mathfrak{g})$ representations. Given two reps V, W, we still have a $U_q(\mathfrak{g})$-action on $V \otimes W$.

However, in this new definition, the obvious map $V \otimes W \to W \otimes V$ is not a map of representations. Luckily, this can be fixed by changing the map a little bit, and multiplying by a formal sum $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ called the "universal R-matrix."

$$b_{V,W} : V \otimes W \xrightarrow{R\cdot} V \otimes W \xrightarrow{\text{flip}} W \otimes V$$

Reshetikhin-Turaev invariants

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Reshetikhin-Turaev invariants

Proposition

The maps $b_{V,W}$ make $U_q(\mathfrak{g})$ into a braided monoidal category.

One way to think about this fact is that if you represent

$b_{V,W} \mapsto$

Then the maps induced by switching factors of big tensor products satisfy the braid relations.

$$(1_W \otimes b_{U,V})(b_{U,W} \otimes 1_V)(1_U \otimes b_{V,W}) \qquad (b_{V,W} \otimes 1_U)(1_V \otimes b_{U,W})(b_{U,V} \otimes 1_W)$$

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On the other hand $b_{V,W} b_{W,V} \neq 1$, as the picture above suggests.

Reshetikhin-Turaev invariants

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Reshetikhin-Turaev invariants

The other important structure on representations of a quantum group is taking dual of representations. As with switching tensor factors, we have to be careful about left and right. There is a contravariant functor $V \mapsto V^*$ called **right dual** (there's also **left dual** which is the same vector space with a different $U_q(\mathfrak{g})$-action).

The category of $U_q(\mathfrak{g})$-representations has canonical maps

- evaluation $V^* \otimes V \to \mathbb{C}(q)$, represented by
- coevaluation $\mathbb{C}(q) \to V \otimes V^*$, represented by

If you want the maps the other way, you need to take left dual.

Not all is lost! After all, we have a map which switches tensor factors. But should we take

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Reshetikhin-Turaev invariants

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Reshetikhin-Turaev invariants

Of course, we can't play favorites. Instead we should take the geometric mean.

If V is irreducible, there's a unique constant $a_V \in \mathbb{C}(q)$ (actually a power of q) such that

$$\frac{1}{\sqrt{a_V}}$$  $$=\quad \sqrt{a_V}$$  .

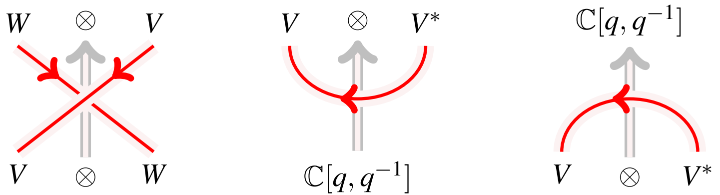
A natural choice of $\sqrt{a_V}$ (I really mean functorial) is called a **ribbon structure**. The reason for the name is that if we interpret the diagrams as drawn with ribbon, then they are  with a left and right twist added, respectively.

Definition

This map is called **quantum trace** *and its vertical flip is called* **quantum cotrace**.

Reshetikhin-Turaev invariants

This allows us to associate a map for any oriented tangle labeled with representations, by associating the braiding to a crossing and appropriate trace or evaluation to cups:



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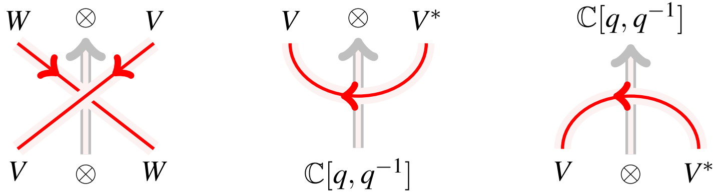

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A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

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Tensor products

In the case of $\mathfrak{sl}_2$, we introduced a graphical calculus for elements of $V_{\underline{\boldsymbol{\lambda}}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$.

- A downward black line on the left means acting by F_i.
- A red line at the left labeled by λ corresponds to $v_\lambda \otimes -$, where v_λ is the highest weight vector of V_λ.

So, we obtain a spanning set of $V_{\underline{\boldsymbol{\lambda}}}$ consisting of vectors like

$F_i(v_{\lambda_1} \otimes F_j v_{\lambda_2}) \leftrightarrow$

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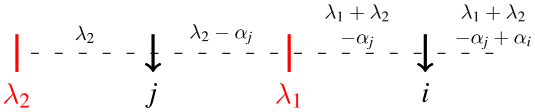
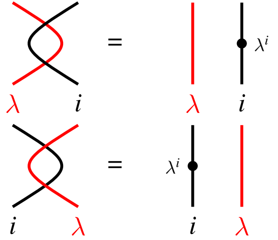
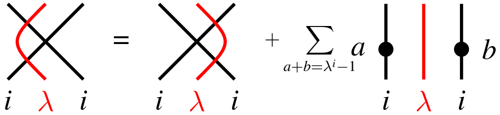


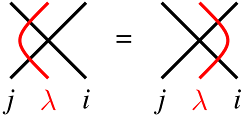
Tensor products

Let $T^{\underline{\boldsymbol{\lambda}}}$ be the algebra whose elements are $\Bbbk$-linear combinations of immersed 1-manifolds with

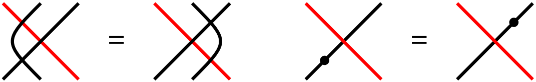
- black components oriented, dotted and labeled with $i \in \Gamma$ and
- red components have no intersections, and are labeled with the weights $\underline{\boldsymbol{\lambda}}$ in order modulo the relations





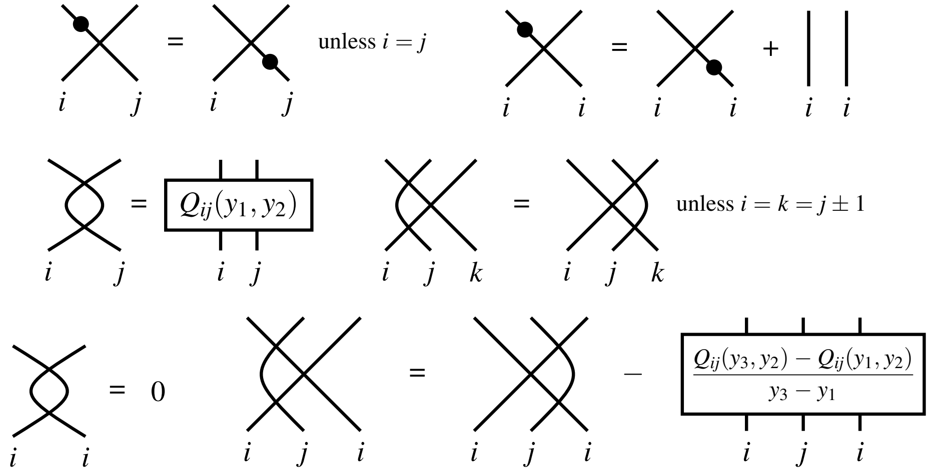


any diagram with a black line at the far left is 0.



and. . . . . .

Diagrams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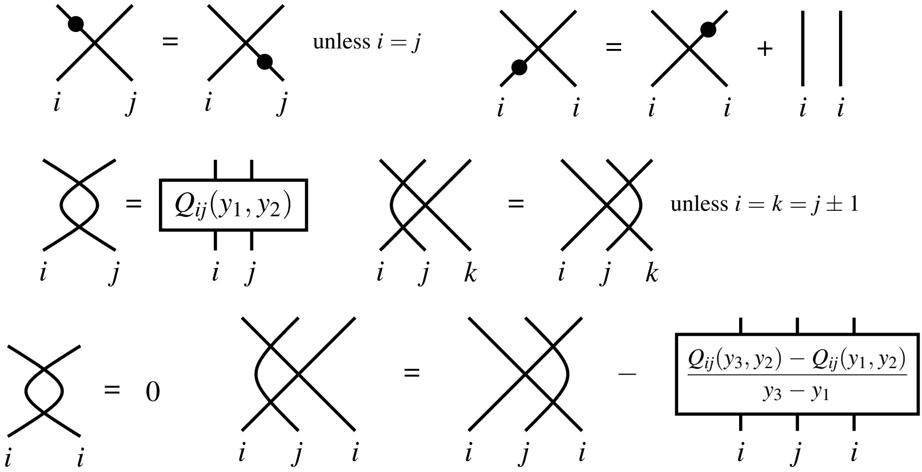
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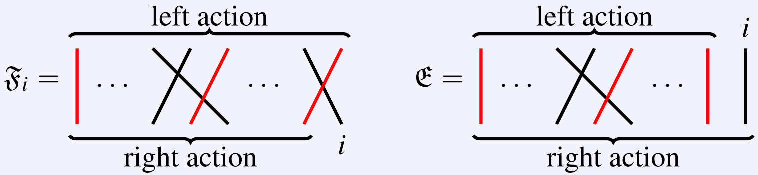
Categorical action

Recall, last time, we defined the notion of a categorical action of $\mathfrak{g}$. For this, we need functors $\mathfrak{F}_i$ and $\mathfrak{E}_i$.

These are induction and restriction functors, which can think of as tensor product with the bimodules:



The action of R_m on the power $\mathfrak{F}^m$ is by attaching pictures at the bottom. Adjunction is essentially automatic.

The tricky part is checking the $\mathfrak{sl}_2$ relations. This is hard.

Grothendieck groups

Theorem

The GG of $T^{\underline{\boldsymbol{\lambda}}}$-pmod *is the Lusztig integral form of* $V_{\underline{\boldsymbol{\lambda}}}$*, sending the functor* $\mathfrak{F}_i$ *to the action of* F_i*, and the functor* $_\lambda$ *(adding a red line) to the inclusion*

$$V \overset{-\otimes v_{high}}{\hookrightarrow} V \otimes V_\lambda.$$

But we'd like to talk about the category $T^{\underline{\lambda}}$-mod, which doesn't have the same Grothendieck group: the map

$$K^0(T^{\underline{\lambda}}\text{-pmod}) \to K^0(T^{\underline{\lambda}}\text{-mod})$$

is injective, but not surjective, since not all simple modules have finite projective resolutions. (Think about $k[x]/(x^2)$).

However, this map is an isomorphism after tensoring with $\mathbb{C}(q)$, so every finite dimensional $T^{\underline{\lambda}}$-module defines a class in $V_{\underline{\lambda}}$.

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Bases

What does the representation theory of this algebra look like?

- Projectives are just summands of the modules $P^{\kappa}_{\mathbf{i}} = T^{\underline{\lambda}} e(\mathbf{i}, \kappa)$ where $e(\mathbf{i}, \kappa)$ is the sequence corresponding to a particular ordering of red and black dots. The indecomposables give you a "canonical basis" (Lusztig's if $\mathfrak{g}$ is symmetric type).
- Simple modules are endowed with a crystal structure (exactly as in Lauda and Vazirani), which is the tensor product of the crystals for V_{λ_i}. These give you a "dual canonical basis."

These objects both give bases of the Grothendieck group which are not very compatible with the tensor product structure. If we're ever going to do any calculations, we're going to need objects that correspond to pure tensors.

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Standard modules

Well, how would we construct the pure tensor $v_1 \otimes F_i v_2$? We have modules corresponding to

$$F_i(v_1 \otimes v_2) = v_1 \otimes F_i v_2 + q^{\lambda^i} F_i v_1 \otimes v_2$$

and

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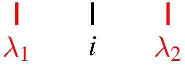

So we'd like to subtract the former from the latter. Of course, in categories you can't subtract, but you can look for submodules. As it happens, the map given by is injective, so modding out by its image gives a module with the right class in the Grothendieck group.

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Standard modules

a "left" crossing

a "right" crossing

Definition

The standard module $S^{\kappa}_{\underline{\boldsymbol{\lambda}}}$ is the quotient of $P^{\kappa}_{\underline{\boldsymbol{\lambda}}}$ by the submodule generated by all diagrams with at least one "left" crossing as above, and no "right" crossings.

Put another way, we can associate a composition to the module $P^{\kappa}_{\mathbf{i}}$ by counting the number of black strands between each pair of reds, and we mod out by the images of all maps from projectives strictly higher in dominance order.

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As you may have guessed

Proposition

$$[S^\kappa_{\mathbf{i}}] = F_{i_{\kappa(1)-1}} \cdots F_{i_1} v_1 \otimes \cdots \otimes F_{i_n} \cdots F_{\kappa(\ell)} v_n$$

This makes standard modules invaluable as "test objects" for functors to see that they behave correctly on the Grothendieck group.

For example, $F_i S^\kappa_{\mathbf{i}}$ has a filtration which categorifies the usual formula

$$\Delta^{(\ell)}(F_i) = F_i \otimes \tilde{K}_i \otimes \cdots \otimes \tilde{K}_i + \cdots + 1 \otimes \cdots \otimes 1 \otimes F_i$$

and similarly for $E_i S^\kappa_{\mathbf{i}}$.

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Unfortunately, if we want to categorify these using the yoga we've used thus far, we run into a problem: **the coefficients aren't positive**.

If you want to have a "direct minus" in a category, you have to use some kind of category of complexes. We let $\mathcal{V}^{\underline{\lambda}}$ be the bounded-above derived category of $T^{\underline{\lambda}}$-mod.

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Braiding and duals

Theorem

Given any sequence $\underline{\boldsymbol{\lambda}}$:

- *For any ℓ-strand braid σ, we have a functor $\mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\sigma\underline{\boldsymbol{\lambda}}}$ which induces the usual braided structure on the GG.*
- *For any $\underline{\boldsymbol{\lambda}}$, and $\underline{\boldsymbol{\lambda}}^+$ given by adding an adjacent pair of dual highest weights, we have functors $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}^+} \to \mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ inducing evaluation and quantum trace on GG, and dually for coevaluation and quantum cotrace (but for a funny ribbon structure!).*

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Braiding

So, now we need to look for braiding functors.

Consider the bimodule $\mathfrak{B}_i$ over $T^{\underline{\boldsymbol{\lambda}}}$ and $T^{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$ given by exactly the same sort of diagrams, but with a single crossing inserted between the ith and $i+1$st crossings.

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The derived tensor product $-\overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}}} \mathfrak{B}_i : \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$ *categorifies the braiding map* $R_i : V_{\underline{\boldsymbol{\lambda}}} \to V_{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$. *The inverse functor is given by* $\mathrm{RHom}(\mathfrak{B}_i, -)$.

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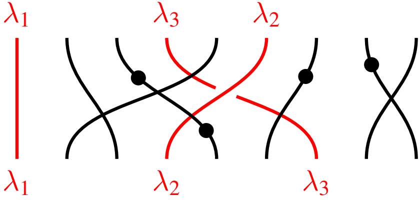

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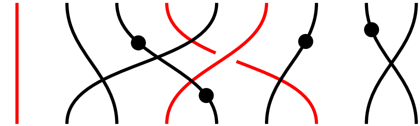

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So, firstly, what does derived tensor product mean? It means, amongst other things, that we could take a projective resolution of $\mathfrak{B}_i$ as a bimodule. This will be a complex in the category $T^{\underline{\lambda}} \otimes T^{\underline{\lambda}}$-pmod which is unique up to homotopy.

Unfortunately, I don't understand at the moment how to write down this complex explicitly. In most cases, it must have infinite length and is quite complex, but it would facilitate computation quite a bit.

On the other hand, part of the magic of homological algebra is that you can figure some things out without knowing this.

Braiding

In particular, how does one check that it actually acts as the braiding? By looking at test objects.

Note that $V_{\lambda_1} \otimes V_{\lambda_2}$ is generated over $U_q(\mathfrak{g})$ by vectors of the form $v \otimes v_{high}$ and under the braiding, these are sent to $q^{?} v_{high} \otimes v$. As we know, these vectors are categorified by standard modules of the form $S_{\mathbf{i}}^{0,n}$.

Proposition

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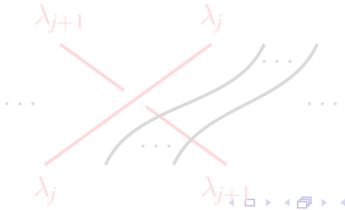

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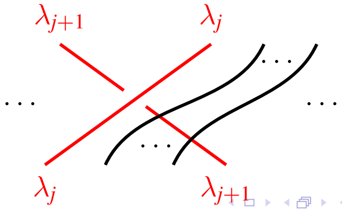

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In particular, how does one check that it actually gives a braid groupoid action?

- The positive and negative twists are inverse because they are adjoint and derived equivalences. (Not easy! Must show that half twist sends projectives to tiltings.)
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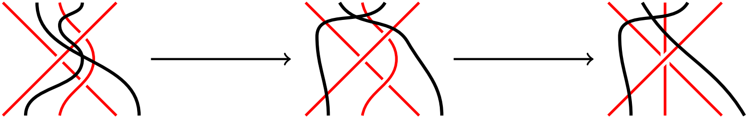

Coevalution and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights λ and $-w_0\lambda = \lambda^*$. We must first define an isomorphism between V_{λ^*} and V_λ^*. That is to say, a pairing $V_\lambda \times V_{\lambda^*} \to \mathbb{C}(q)$.

We start with a chosen highest weight vector of both representations v_λ, v_{λ^*} (this comes from the irrep in T_λ^λ-mod $\cong \mathbf{k}$-mod). So, a pairing is fixed by a choice of lowest weight vector.

Pick a reduced expression

$$w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.$$

Then we have a lowest weight vector of the form

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Invariants

We should look for a categorification of the unique invariant vector $c \in V_\lambda \otimes V_{\lambda^*}$. We can actually guess quite easily what this should be.

The space of invariants is orthogonal under the Euler form to all projectives of the form $\mathcal{F}_i M$ for any i. We know by counting arguments that all but one indecomposable projective is a summand of a $\mathcal{F}_i M$.

We actually know exactly what this remaining projective P_λ is; it corresponds to the sequence of weights and roots

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- The coevaluation functor is categorified by the functor $\mathcal{V}^{\emptyset} \cong \mathsf{Vect} \to \mathcal{V}^{\lambda,\lambda^*}$ sending $\mathbb{C} \to L_\lambda$.
- The evaluation functor is categorified by

$$\mathrm{RHom}(L_\lambda, -)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)\colon \mathcal{V}^{\lambda,\lambda^*} \to \mathcal{V}^{\emptyset} \cong D_{\mathsf{fd}}(\mathsf{Vect}).$$

Now, we know that if we want quantum trace, we should compromise between

$L_\lambda[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$ and $L_\lambda[-2\rho^\vee(\lambda)](-2\langle\lambda,\rho\rangle)$

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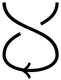

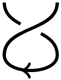

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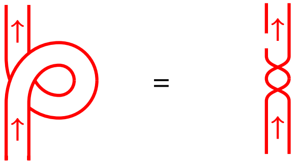
Ribbon structure

So this decategorifies to $(-1)^{2\rho^\vee(\lambda)} q^{2\langle \lambda,\rho\rangle}$. Note: this is a strange ribbon element! (It appeared in work of Snyder and Tingley on half-twist elements.)

For each ribbon element, there is a notion of "quantum dimension," and in this picture, $\mathrm{qdim} V|_{q=1} = (-1)^{2\rho^\vee(\lambda)} \dim V$. For example, in $\mathfrak{sl}_2$,

$$\mathrm{qdim} V_n = (-1)^n \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}.$$

From now on, all my knots are ribbon knots (in the blackboard framing), and I'll really get invariants of ribbon knots (but twists just give grading shifts).



Coevalution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded Euler characteristic given by the quantum dimension of V_λ.

If V_λ is miniscule, then everything works beautifully. The dimension of A_λ is really the dimension of V_λ. In particular, if $\lambda = \omega_i$ for $\mathfrak{g} = \mathfrak{sl}_n$, then $A_\lambda \cong H^*(\mathsf{Grass}(i,n))$.

Conjecture

If λ is miniscule, $A_\lambda \cong H^*(\mathsf{Gr}_\lambda)$.

On the other hand, if λ is not miniscule, things blow up. For example, if $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda = 2$, then

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$$\sum_{i,j}(-t)^j \dim_q A^j_\lambda \neq q^{-2}t^2 + 1 + q^2t^{-2}$$

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$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded Euler characteristic given by the quantum dimension of V_λ.

If V_λ is miniscule, then everything works beautifully. The dimension of A_λ is really the dimension of V_λ. In particular, if $\lambda = \omega_i$ for $\mathfrak{g} = \mathfrak{sl}_n$, then $A_\lambda \cong H^*(\mathsf{Grass}(i,n))$.

Conjecture

If λ is miniscule, $A_\lambda \cong H^*(\mathsf{Gr}_\lambda)$.

On the other hand, if λ is not miniscule, things blow up. For example, if $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda = 2$, then

$$\textstyle\sum_{i,j}(-t)^j \dim_q A_\lambda^j = q^{-2}t^2 + 1 + q^2t^{-2} + \frac{q^2-q^2t}{1-t^2q^4}$$

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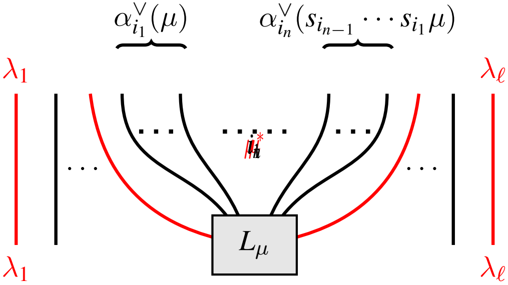
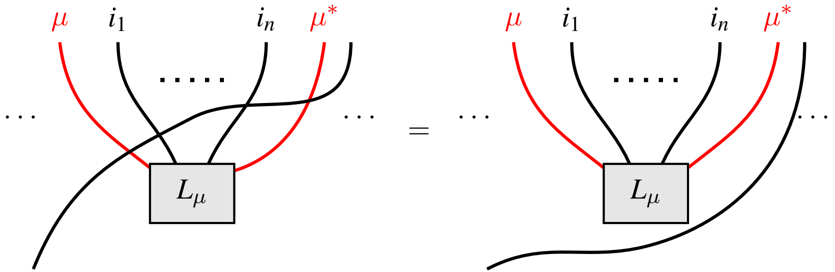


Coevalution and quantum trace

There's exactly one interesting relation here, which says that



$$F_i v \otimes c_\lambda = F_i(v \otimes c_\lambda).$$

Theorem

Tensor product with this bimodule categorifies coevaluation/quantum cotrace, and Hom with it categorifies evaluation/quantum trace.

Knot invariants

Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.

For a link L, we get a functor $F_L : \mathcal{V}^\emptyset \cong D(\mathsf{Vect}) \to \mathcal{V}^\emptyset \cong D(\mathsf{Vect})$. So $F_L(\mathbb{C})$ is a complex of vector spaces (actually graded vector spaces).

Theorem

The cohomology of $F_L(\mathbb{C})$ is a knot invariant, and finite-dimensional in each homological and each graded degree. The graded Euler characteristic of this complex is $J_{V,L}(q)$.

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Knot invariants

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

Start with $\mathbb{C}$.

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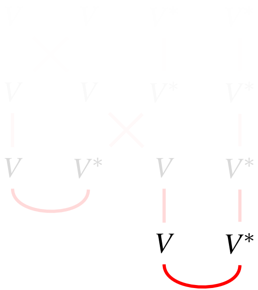

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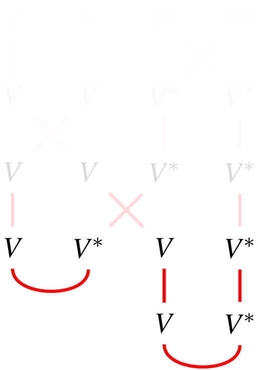

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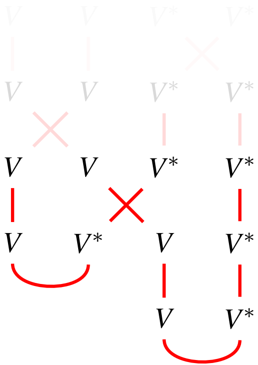

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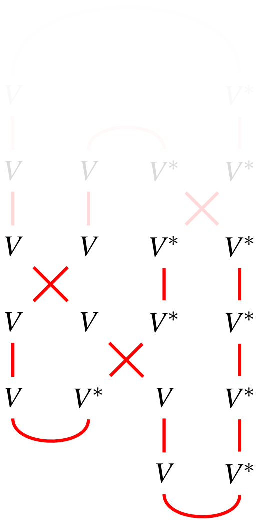

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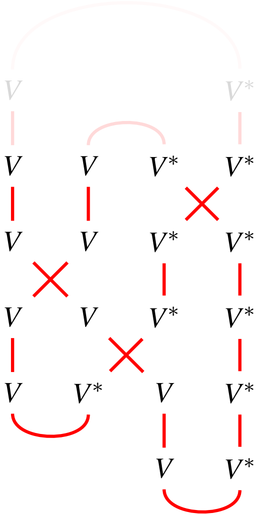

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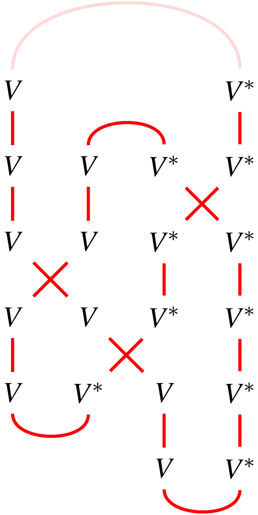

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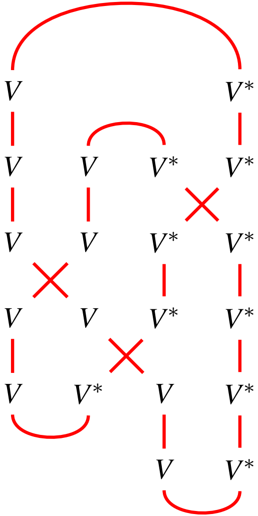

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4d TQFT

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The quantum knot invariants arise from a 3-d TQFT: Chern-Simons theory. You can think of this as built up from attaching the category of $U_q(\mathfrak{g})$ representation to a circle and building the 2-and 3-dimensional layers from that.

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Next time

Next time I'll talk about how to relate this construction to the other ones I've mentioned, especially those of Khovanov-Rozansky and Cautis-Kamnitzer.

Doing that will also require some discussion of connections to the geometry of quiver varieties.