Higher representation theory in algebra and geometry: Lecture IX

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UVA

April 10, 2014

References

For this lecture, useful references include:

- B.W., *Knot invariants and higher representation theory*

The slides for the talk are on my webpage at:

`http://people.virginia.edu/~btw4e/lecture-8.pdf`

You can also find some proofs that I didn't feel like going through in class at:

`https://pages.shanti.virginia.edu/Higher_Rep_Theory/`

Roadmap

$$\begin{tikzcd}
\text{quantum groups } U_q(\mathfrak{g}) \arrow[d] \\
\text{ribbon category of } U_q(\mathfrak{g})\text{-reps} \arrow[d] \\
\text{quantum knot polynomials} \\[-20pt]
\text{(Jones polynomial, etc.)}
\end{tikzcd}$$

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 & \text{2-categories } \mathcal{U} \arrow[ul, dashed] \arrow[d] \\
\text{ribbon category of } U_q(\mathfrak{g})\text{-reps} \arrow[d] & \boxed{???} \arrow[l, dashed] \arrow[d] \\
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Braiding and duals

Theorem

Given any sequence $\underline{\boldsymbol{\lambda}}$:

- *For any ℓ-strand braid σ, we have a functor $\mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\sigma\underline{\boldsymbol{\lambda}}}$ which induces the usual braided structure on the GG.*
- *For any $\underline{\boldsymbol{\lambda}}$, and $\underline{\boldsymbol{\lambda}}^+$ given by adding an adjacent pair of dual highest weights, we have functors $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}^+} \to \mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ inducing evaluation and quantum trace on GG, and dually for coevaluation and quantum cotrace (but for a funny ribbon structure!).*

My goal for the rest of this talk is to describe these functors, and how they give knot invariants.

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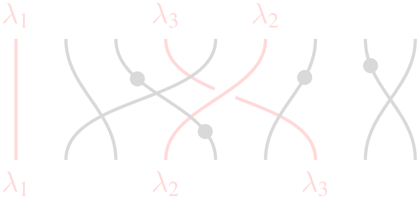
Braiding

So, now we need to look for braiding functors.

Consider the bimodule $\mathfrak{B}_i$ over $T^{\underline{\boldsymbol{\lambda}}}$ and $T^{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$ given by exactly the same sort of diagrams, but with a single crossing inserted between the ith and $i+1$st crossings.



Theorem

The derived tensor product $\mathfrak{B}_i \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}}} - : \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$ *categorifies the braiding map* $R_i : V_{\underline{\boldsymbol{\lambda}}} \to V_{(i,i+1)\cdot\underline{\boldsymbol{\lambda}}}$. *The inverse functor is given by* $\mathrm{RHom}(\mathfrak{B}_i, -)$.

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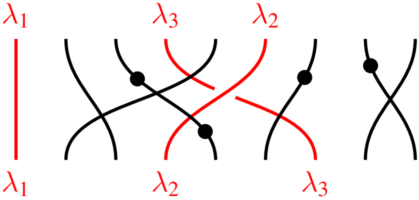

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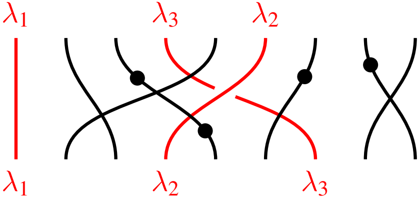

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Braiding

In order to understand these bimodules, we have to prove a number of facts:

Lemma

The bimodules $\mathfrak{B}_i$ satisfy the properties:

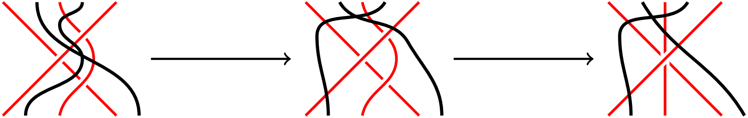
- *the tensor products $\mathfrak{B}_\sigma = \mathfrak{B}_{i_1} \otimes \cdots \otimes \mathfrak{B}_{i_m}$ for a reduced word in S_ℓ on the product $\sigma = s_{i_1} \cdots s_{i_n}$ (that is, any two such reduced words give canonically isomorphic bimodules).*
- *the bimodule $\mathfrak{B}_\sigma$ is standard filtered as a left and right module (thus sends projectives to standard filtered modules and standards to modules).*
- *the functors $\mathfrak{B}_{w_0} \otimes -$ sends projectives to tiltings (and tiltings to injectives).*

Corollary

The functors $\mathfrak{B}_\sigma \overset{L}{\otimes} -$ are derived equivalences that define a strong action of the braid group on the categories $\oplus_{w \in S_\ell} \mathcal{V}^{w\underline{\boldsymbol{\lambda}}}$.

Braiding

The proof that the braid relations hold is very pictorial (though of course, you need to do a lot of homological algebra to see that it means what you think it does):



Coevalution and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights λ and $-w_0\lambda = \lambda^*$. We must first define an isomorphism between V_{λ^*} and V_λ^*. That is to say, a pairing $V_\lambda \times V_{\lambda^*} \to \mathbb{C}(q)$.

We start with a chosen highest weight vector of both representations v_λ, v_{λ^*} (this comes from the irrep in T_λ^λ-mod $\cong \Bbbk$-mod). So, a pairing is fixed by a choice of lowest weight vector.

Pick a reduced expression

$$w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.$$

Then we have a lowest weight vector of the form

$$v_{low} = F_{i_n}^{(\alpha_n^\vee(s_{n-1}\cdots s_1\lambda))} \cdots F_{i_2}^{(\alpha_2^\vee(s_1\lambda))} F_{i_1}^{(\alpha_1^\vee(\lambda))} v_\lambda$$

We will always choose this one.

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Invariants

We should look for a categorification of the unique invariant vector $c \in V_\lambda \otimes V_{\lambda^*}$. We can actually guess quite easily what this should be.

The space of invariants is orthogonal under the Euler form to all projectives of the form $\mathcal{F}_i M$ for any i. We know by counting arguments that all but one indecomposable projective is a summand of a $\mathcal{F}_i M$.

We actually know exactly what this remaining projective P_λ is; it corresponds to the sequence of weights and roots

$$(\lambda, \alpha_1^{(\alpha_1^\vee(\lambda))}, \alpha_2^{(\alpha_2^\vee(s_1\lambda))}, \dots, \alpha_n^{(\alpha_n^\vee(s_{n-1}\cdots s_1\lambda))}, \lambda^*).$$

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Coevalution and evaluation

- The coevaluation functor is categorified by the functor $\mathcal{V}^{\emptyset} \cong \mathsf{Vect} \to \mathcal{V}^{\lambda,\lambda^*}$ sending $\mathbb{C} \to L_\lambda$.
- The evaluation functor is categorified by

$$\mathrm{RHom}(L_\lambda, -)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle) \colon \mathcal{V}^{\lambda,\lambda^*} \to \mathcal{V}^{\emptyset} \cong D_{\mathsf{fd}}(\mathsf{Vect}).$$

Now, we know that if we want quantum trace, we should compromise between

$L_\lambda[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$ and $L_\lambda[-2\rho^\vee(\lambda)](-2\langle\lambda,\rho\rangle)$

Definition

The positive ribbon twist acts on the category by $[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$.

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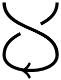

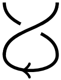

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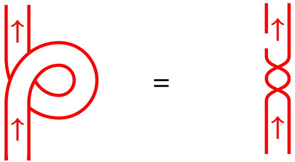
Ribbon structure

So this decategorifies to $(-1)^{2\rho^\vee(\lambda)}q^{2\langle\lambda,\rho\rangle}$. Note: this is a strange ribbon element! (It appeared in work of Snyder and Tingley on half-twist elements.)

For each ribbon element, there is a notion of "quantum dimension," and in this picture, $\mathrm{qdim} V|_{q=1} = (-1)^{2\rho^\vee(\lambda)}\dim V$. For example, in $\mathfrak{sl}_2$,

$$\mathrm{qdim} V_n = (-1)^n \frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}.$$

From now on, all my knots are ribbon knots (in the blackboard framing), and I'll really get invariants of ribbon knots (but twists just give grading shifts).



Coevalution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded Euler characteristic given by the quantum dimension of V_λ.

If V_λ is miniscule, then everything works beautifully. The dimension of A_λ is really the dimension of V_λ. In particular, if $\lambda = \omega_i$ for $\mathfrak{g} = \mathfrak{sl}_n$, then $A_\lambda \cong H^*(\mathsf{Grass}(i,n))$.

Conjecture

If λ is miniscule, $A_\lambda \cong H^*(\mathsf{Gr}_\lambda)$.

On the other hand, if λ is not miniscule, things blow up. For example, if $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda = 2$, then

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$$\textstyle\sum_{i,j}(-t)^j \dim_q A^j_\lambda \neq q^{-2}t^2 + 1 + q^2t^{-2}$$

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$$\textstyle\sum_{i,j}(-t)^j \dim_q A_\lambda^j = q^{-2}t^2 + 1 + q^2t^{-2} + \frac{q^2-q^2t}{1-t^2q^4}$$

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Coevalution and quantum trace

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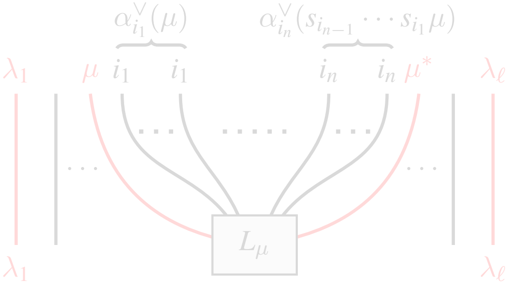

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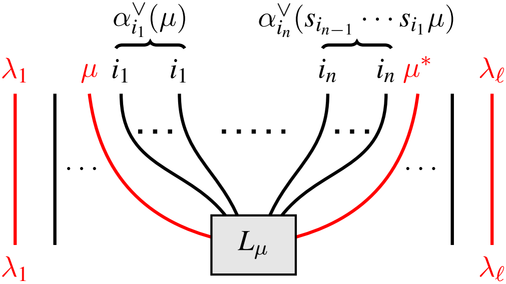
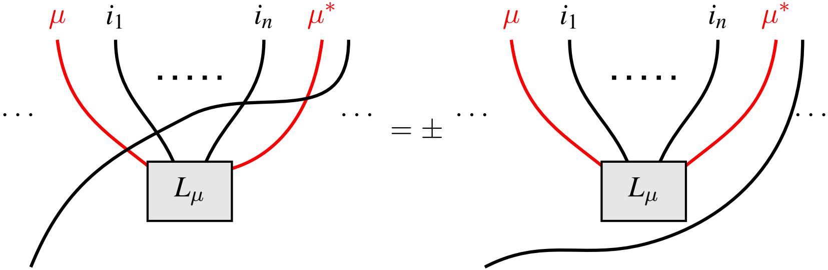


Coevalution and quantum trace

There’s exactly one interesting relation here, which says that



$$F_i v \otimes c_\lambda = F_i(v \otimes c_\lambda).$$

Theorem

Tensor product with this bimodule categorifies coevaluation/quantum cotrace, and Hom with it categorifies evaluation/quantum trace.

Knot invariants

Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.

For a link L, we get a functor $F_L : \mathcal{V}^{\emptyset} \cong D(\mathsf{Vect}) \to \mathcal{V}^{\emptyset} \cong D(\mathsf{Vect})$. So $F_L(\mathbb{C})$ is a complex of vector spaces (actually graded vector spaces).

Theorem

The cohomology of $F_L(\mathbb{C})$ is a knot invariant, and finite-dimensional in each homological and each graded degree. The graded Euler characteristic of this complex is $J_{V,L}(q)$.

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Knot invariants

$A_3 = \mathrm{RHom}(\mathfrak{B}, B_2) \longrightarrow$ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2} \longrightarrow$ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2} \longrightarrow$ Replace with projective resolution B_1

Start with $\mathbb{C}$.

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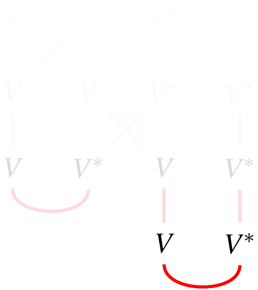

$A_4 = B_3 \otimes \mathfrak{B}_1 \longrightarrow$ Replace with projective resolution B_4

$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2) \longrightarrow$ Replace with projective resolution B_3

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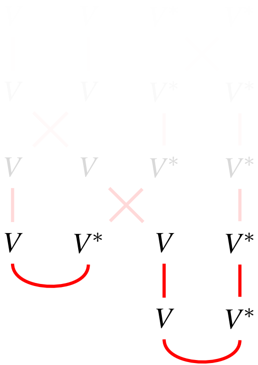

$A_5 = B_4 \otimes \mathfrak{B}_3 \longrightarrow$ Replace with injective resolution B_5

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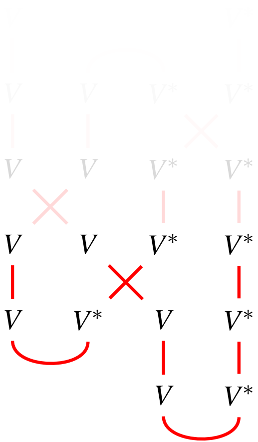

$A_6 = \mathrm{RHom}(\mathfrak{K}_V^{2,3}, B_5) \longrightarrow$ Replace with injective resolution B_6

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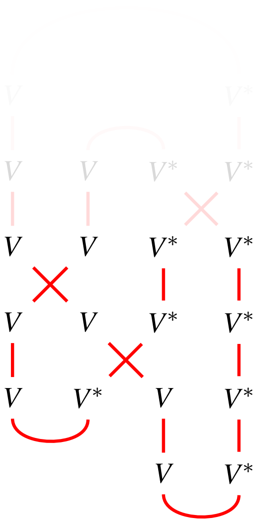
Start with $\mathbb{C}$.

Knot invariants



$A_7 = \mathrm{RHom}(\mathfrak{K}_V^{1,2}, B_6) \longrightarrow$ Knot homology!

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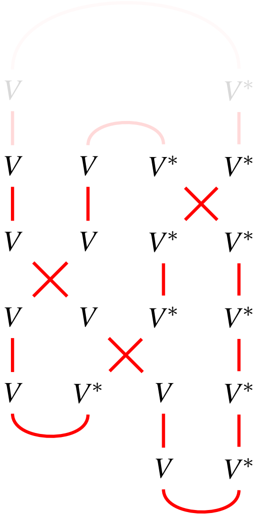

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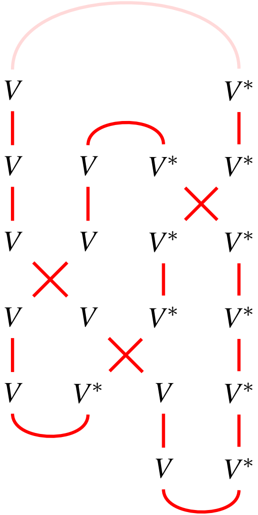

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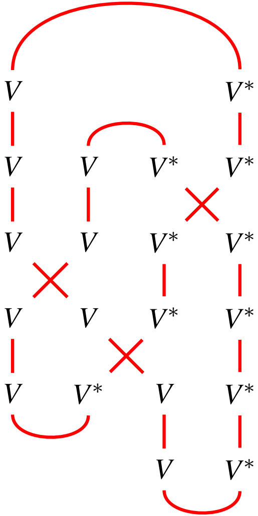

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4d TQFT

One of the inspirations for studying categorifications is the connections between higher categories and quantum field theory.

The quantum knot invariants arise from a 3-d TQFT: Chern-Simons theory. You can think of this as built up from attaching the category of $U_q(\mathfrak{g})$ representation to a circle and building the 2-and 3-dimensional layers from that.

Can one make a 4-dimensional TQFT of some kind out the category of 2-representations of this categorified quantum group?

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The case of $\mathfrak{sl}_n$

In the case of $\mathfrak{sl}_2$, we saw that this complicated bimodule broke into two "simpler" pieces that allow us to write it as a cone:

One might hope that there is a similar decomposition for $\mathfrak{sl}_n$; unfortunately, the complicatedness of this decomposition is related to the number of summands in $\bigwedge^i \mathbb{C}^n \otimes \bigwedge^j \mathbb{C}^n$.

We'll only be interested in fundamental case $\underline{\boldsymbol{\lambda}} = (\omega_{p_1}, \dots, \omega_{p_\ell})$; let $T^{\mathbf{p}}_{\mathbf{q}}$ be associated algebra, with weight fixed to be $\mathbf{q} = (q_1, \dots, q_n)$.

Relation to category $\mathcal{O}$

One way to get to know the $\mathfrak{sl}_n$ picture better is to compare it to category $\mathcal{O}$ for $\mathfrak{gl}_p$.

Theorem (W.)

There is an equivalence of categories from $T^{\mathbf{p}}_{\mathbf{q}}$-mod *to the block* $\mathcal{O}^{\mathbf{p}}_{\mathbf{q}}$ *of parabolic category* $\mathcal{O}$ *for* $\mathfrak{gl}_p$ *for the parabolic of* **p***-block upper triangular matrices containg the Verma module of highest weight* $(1+1, 1+2, \dots, 1+q_1, 2+q_1+1, \dots 2+q_1+q_2, \dots, n+p-q_n+1, \dots, n+p)$. *This equivalence intertwines:*

- *categorification functors with translation functors,*
- *braiding functors with twisting functors*
- *cups and caps with Zuckerman functors (composed with Enright-Shelton equivalences)*

and thus matches our homology with Mazorchuk-Stroppel-Sussan homology.

Relation to category $\mathcal{O}$

The proof is actually fairly straightforward. Of course, we must find a projective generator of this block of $\mathcal{O}^{\mathbf{p}}$ whose endomorphism algebra is $T^{\mathbf{p}}_{\mathbf{q}}$.

Let $\mathfrak{p}_{\mathbf{p}}$ be the $\mathbf{p}$-block upper triangular matrices, and let $V_{(i)}$ be the *i*th step of the natural filtration on $\mathbb{C}^n$. Let $\mathrm{pr}_{\mathbf{q}}$ be the projection to the corresponding block in $\mathcal{O}^{\mathbf{p}}$.

Theorem (Brundan-Kleshchev)

The module $P = \bigoplus_{\mathbf{a}} \mathrm{pr}_{\mathbf{q}}(U(\mathfrak{gl}_p) \otimes_{U(\mathfrak{p}_{\mathbf{p}})} (V^{a_1}_{(1)} \otimes \cdots \otimes V^{a_\ell}_{(\ell)}))$ *is a projective generator of* $\mathcal{O}^{\mathbf{p}}_{\mathbf{q}}$ *with* $\mathrm{End}(P) \cong T^{\mathbf{p}}_{\mathbf{q}}$.

The proof uses “higher Schur-Weyl duality” and should be thought of as a generalization of the isomorphism between the degenerate affine Hecke algebra and KLR algebra in type A.

Koszulity

Knowing this has some beautiful consequences:

Theorem

The algebra $T^{\underline{\boldsymbol{\lambda}}}$ is Koszul. Thus $D^b(T^{\underline{\boldsymbol{\lambda}}}\text{-gmod})$ has a Koszul dual t-structure, whose heart is linear complexes of projectives.

Every indecomposable projective has a prefered grading (the one where $P \cong \mathrm{Hom}(P, T^{\underline{\boldsymbol{\lambda}}})$), and a complex of projectives is **linear** if the ith degree is shifted by $-i$ from its prefered grading.

Theorem

The (graded abelian) category of linear complexes of projectives over $T_{\mathbf{q}}^{\mathbf{p}}$ is equivalent to $T_{\mathbf{p}}^{w_0\mathbf{q}}\text{-gmod}$. That is, $D^b(T_{\mathbf{p}}^{w_0\mathbf{q}}\text{-gmod}) \cong D^b(T_{\mathbf{q}}^{\mathbf{p}}\text{-gmod})$.

Note ℓ and n have switched places: $T_{\mathbf{p}}^{\mathbf{q}}$-gmod categorifies a tensor product of n representations of $\mathfrak{sl}_{\ell}$.

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Koszul duality

Now, consider the sum $\bigoplus_{|\mathbf{p}|=|\mathbf{q}|=p} D^b(T^{\mathbf{p}}_{\mathbf{q}})$ for fixed n, ℓ, p.

Theorem

This categorifies $\bigwedge^p(\mathbb{C}^\ell \otimes \mathbb{C}^n) \cong \oplus_{\mathbf{p}} \bigwedge^{p_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{p_\ell} \mathbb{C}^n$, with the obvious categorical action of $\mathfrak{sl}_n$ and a commuting action of $\mathfrak{sl}_\ell$ coming from Koszul duality. These are each others "full commutants" if one is willing to take cones.

I should note that all of these are gotten by transporting results of Beilinson, Ginzburg, Soergel, Bäckelin, Stroppel, etc. from category $\mathcal{O}$. One can prove them directly from the diagrammatics (joint work in progress with M. Mackaay).

The internal braid group

In particular, the braiding functor corresponds to the image of *some* complex of functors in the categorification of $\mathfrak{sl}_\ell$.

Recall way back when we talked about derived equivalences for opposite weight spaces in $\mathfrak{sl}_2$. Well, now we have a bunch of different $\mathfrak{sl}_2$ actions; switching their weights generates the Weyl group.

Let $\Theta_i = \cdots \to \mathcal{F}^{(n+k)}\mathcal{E}^{(k)} \longrightarrow \mathcal{F}^{(n+k-1)}\mathcal{E}^{(k-1)} \to \cdots$ be the Chuang-Rouquier complex and its associated functor on homotopy/derived categories.

Theorem

The functors Θ_i satisfy the Artin braid relations (i.e. the relations of the Weyl group that preserve length). In particular, if $\mathfrak{g} = \mathfrak{sl}_n$, they generate a copy of the usual braid group on n strands.

The internal braid group

Theorem

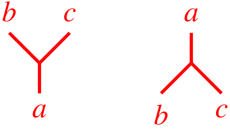
The action of the internal braid group of $\mathfrak{sl}_\ell$ *agrees with the braiding functors for* $\mathfrak{sl}_n$ *(up to shift).*

This has the enormous advantage of removing the need to compute a projective resolution of everything at every step (though one still does have a complex from every crossing).

Intertwiners

As I discussed in my talk yesterday, we can also identify this dual action with certain webs.

For $a \equiv b + c \pmod{n}$, we let the diagrams



denote the unique non-zero maps (with scalars normalized carefully)

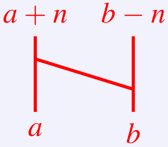
$$\bigwedge^a V \to \bigwedge^b V \otimes \bigwedge^c V \qquad \bigwedge^b V \otimes \bigwedge^c V \to \bigwedge^a V$$

Intertwiners

Proposition

The action of $U_q(\mathfrak{sl}_\ell)$ *on* $\bigoplus_{p_1+\cdots+p_\ell=p} \bigwedge^{p_1} V \otimes \cdots \otimes \bigwedge^{p_\ell} V$ *sends*

$E_i^{(n)} \mapsto$



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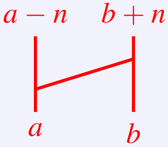
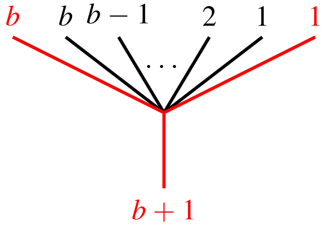


In fact, we can describe the dual categorical action of $\mathfrak{sl}_\ell$ in terms of functors attached to webs.

Intertwiners

Exactly what this bimodule should be in general is a little tricky, so let me just focus on the case where b or c is 1. In that case, one takes diagrams and relations exactly as before, except that at one spot, one includes the picture



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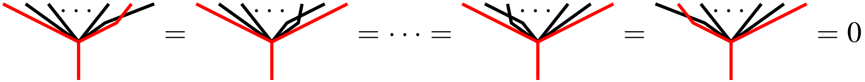

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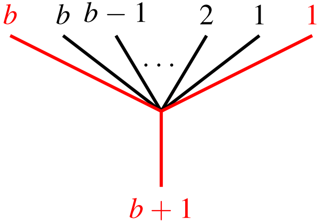

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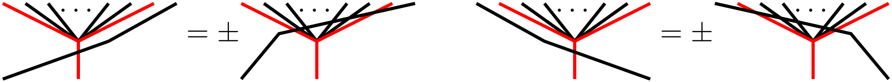

Knot homology from the internal braid group action

You can think (for informal purposes as the moment) that the 2-morphisms in $\mathcal{U}$ are given by certain singular cobordisms between these webs.

One would like to formalize this statement by attaching a natural transformation to each "foam" (a certain kind of singular cobordism). This will likely be possible once we learn the correct relations these should satisfy, which Queffelec and Rose should be telling me any day now.

One this is done, functoriality for $\mathfrak{sl}_n$ should fall out immediately.

Knot homology from the internal braid group action

As I mentioned in my talk yesterday, you can actually rephrase this construction purely in terms of the $\mathfrak{sl}_\ell$-action, with no need to actually know about the tensor products.

One natural way to get a knot from a braid is **plait closure**; if you have a braid σ with strands labeled with wedge powers of $\mathbb{C}^n$, then you can close it up with nested cups at top and bottom if the top labels $(a_1, \dots, a_{2\ell})$ and bottom labels $(b_1, \dots, b_{2\ell})$ satisfy

$$a_i + a_{2\ell-i} = b_i + b_{2\ell-i} = n.$$

Theorem (Cautis)

For each sequence $(a_1, \cdots, a_{2\ell})$ with $a_i + a_{2\ell-i} = n$, there is a distinguished indecomposable projective $P_{\mathbf{a}}$ over $T^{\ell n}$ such that $\operatorname{Hom}(\Theta_\sigma P_{\mathbf{a}}, P_{\mathbf{b}})$ is the $\mathfrak{sl}_n$-homology of the plait closure of σ.

Our construction is the image of this one under Koszul duality.